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# *Localized Reduced Basis for Neutron Diffusion*

**F. A. B. Silva**

TEXAS A&M UNIVERSITY
Department of Nuclear Engineering

# Eigenvalue Problem – Strong Formulation

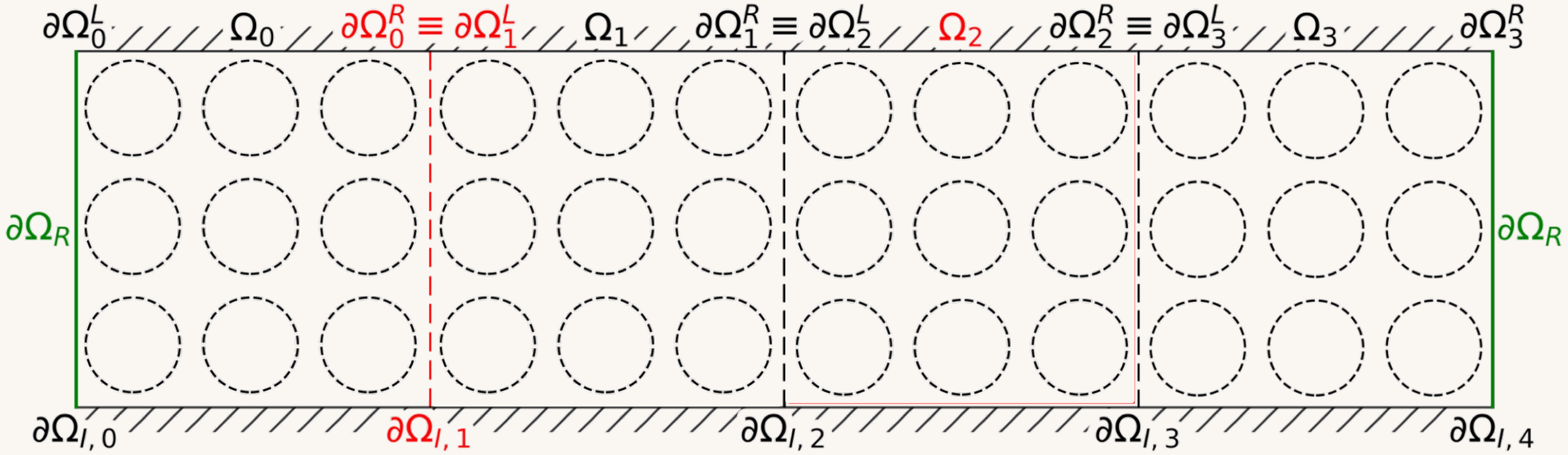
Consider the neutron diffusion generalized eigenvalue problem:
Find the dominant eigen-couple $\phi : \Omega \to \mathbb{R}$, $\mathrm{k}_{eff} \in \mathbb{R}^{+}$, with $\|\phi\|_{\mathrm{L}^2(\Omega)} = 1$, such that

$$\begin{cases} -\nabla \cdot (D_c \nabla \phi) + \Sigma_a \phi = 1/\mathrm{k}_{\mathrm{eff}} \Sigma_f \phi & \text{in } \Omega, \\ -D_c \nabla \phi \cdot \mathbf{n} = \varrho \phi & \text{on } \partial\Omega_{\mathrm{R}}, \\ -D_c \nabla \phi \cdot \mathbf{n} = 0 & \text{on } \partial\Omega_{\mathrm{N}}, \end{cases}$$

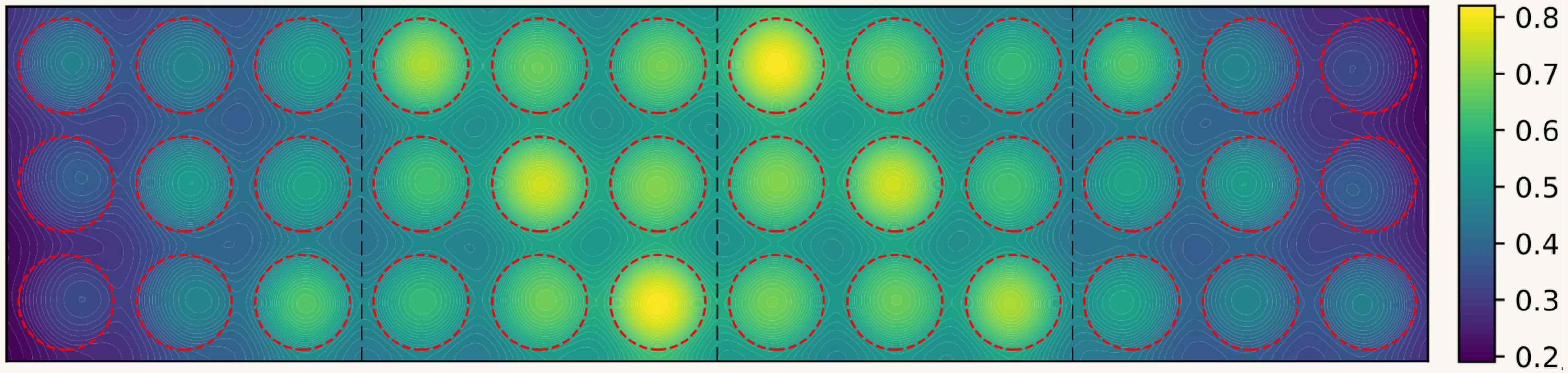
where:

- $\Omega \subset \mathbb{R}^d, d \in \{2,3\}$ is a compact and decomposable Lipschitz domain
- $\partial\Omega = \partial\Omega_{\mathrm{R}} \cup \partial\Omega_{\mathrm{N}}$, $\partial\Omega_{\mathrm{R}} \cap \partial\Omega_{\mathrm{N}} = \emptyset$ is the outer boundary of the domain
- $D_c, \Sigma_a, \Sigma_f : \Omega \to \mathbb{R}^{+}$ are heterogeneous material properties (with $D_c, \Sigma_a \geq 0$)
- $\varrho : \partial\Omega_R \to \mathbb{R}^{+}$ is the Robin boundary coefficient, or albedo
- $\mathbf{n} \in \mathcal{S}^{d-1}$ is the outward unit normal to the outer boundary

# Reference Problem




$\phi_h =$




$k_{\text{eff}} = 0.9905$

$N_h = 23{,}248$

# Eigenvalue Problem – Weak Formulation

For a conforming finite-dimensional function space $\mathcal{V}_h(\Omega) \subset H^1(\Omega)$, the discrete weak form of (1) reads as follows: Find the dominant eigen-couple $\phi_h, \mathrm{k}_{eff} \in \mathcal{V}_h(\Omega) \times \mathbb{R}^+$, with $\|\phi_h\|_{\mathrm{L}^2(\Omega)} = 1$, such that

$$\int_\Omega D_c \nabla\phi_h \cdot \nabla\xi_h \, d\Omega + \int_\Omega \Sigma_a \, \phi_h \, \xi_h \, d\Omega + \int_{\partial\Omega_\mathrm{R}} \varrho \, \phi_h \, \xi_h \, d\gamma = \frac{1}{\mathrm{k}_{eff}} \int_\Omega \Sigma_f \, \phi_h \, \xi_h \, d\Omega \,, \qquad \forall \xi_h \in \mathcal{V}_h(\Omega),$$

or alternatively, introducing bilinear forms $a^\Omega, b^\Omega : \mathcal{V}_h(\Omega) \times \mathcal{V}_h(\Omega) \to \mathbb{R}^+$:
Find the dominant eigen-couple $\phi_h, \mathrm{k}_{eff} \in \mathcal{V}_h(\Omega) \times \mathbb{R}^+$, with $\|\phi_h\|_{\mathrm{L}^2(\Omega)} = 1$, such that

$$b^\Omega(\phi_h, \xi_h) = \mathrm{k}_{eff} \, a^\Omega(\phi_h, \xi_h), \qquad \forall \xi_h \in \mathcal{V}_h(\Omega),$$

where for all $\eta_h, \xi_h \in \mathcal{V}_h(\Omega)$

$$a^\Omega(\eta_h, \xi_h) := \int_\Omega D_c \nabla\eta_h \cdot \nabla\xi_h d\Omega + \int_\Omega \Sigma_a \, \eta_h \, \xi_h \, d\Omega + \int_{\partial\Omega_\mathrm{R}} \varrho \, \eta_h \, \xi_h \, d\gamma \,,$$

$$b^\Omega(\eta_h, \xi_h) := \int_\Omega \Sigma_f \, \eta_h \, \xi_h \, d\Omega \,.$$

# Eigenvalue Problem – Variational Formulation

The same generalized eigenvalue problem can alternatively be written in variational form as

$$(\phi_h, \mathrm{k}_{\mathrm{eff}}) = \underset{\substack{\eta_h \in \mathcal{V}_h(\Omega) \\ \|\eta_h\|_{\mathrm{L}^2(\Omega)}=1}}{\arg\max} \frac{b^\Omega(\eta_h, \eta_h)}{a^\Omega(\eta_h, \eta_h)}.$$

With this work, we aim at solving this problem on a local reduced basis space $\mathcal{V}_\varepsilon(\Omega) = \bigoplus_{i=1}^{N_D} \mathcal{V}_{\varepsilon,i}(\Omega) \subset \mathcal{V}_h(\Omega)$ exploiting the decomposability of the global domain. For the construction of $\mathcal{V}_\varepsilon(\Omega)$, we proceed assuming

1. $\phi_h, \mathrm{k}_{\mathrm{eff}}$ are both known
2. $\mathrm{k}_{\mathrm{eff}}$ is known, $\phi_h$ is unknown
3. $\phi_h, \mathrm{k}_{\mathrm{eff}}$ are both unknown

# Domain Decomposition

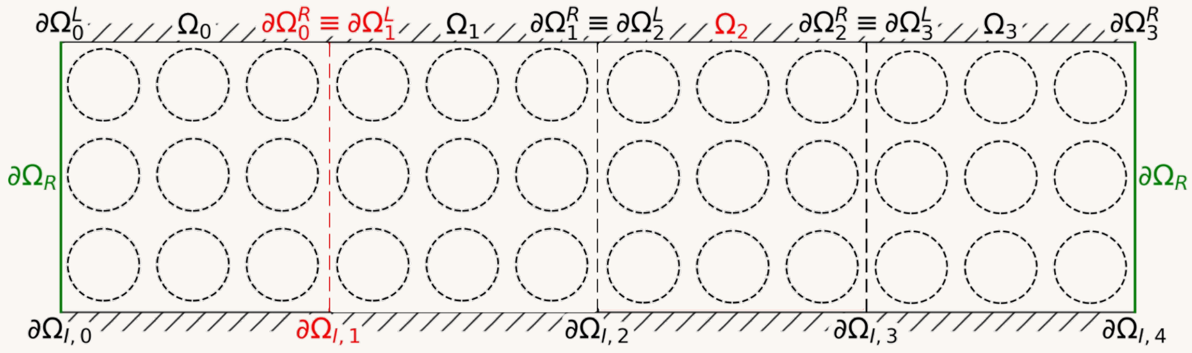
We assume the global domain $\Omega$ can be decomposed in $N_D \in \mathbb{N}$ non-overlapping subdomains $\Omega_\mathrm{i}$, such that

$\Omega = \bigcup_{i=1}^{N_D} \Omega_i$, with $\partial\Omega_\mathrm{i} \cap \partial\Omega_\mathrm{j} = \emptyset$ for all $\mathrm{i}, \mathrm{j} \in \{1, \dots, N_D\}$.

For every subdomain $\Omega_i$, we define an oversampling region $\Omega_i \subset \Omega_i^+ \subset \Omega$, such that $\partial\overline{\Omega}_i^+ \cap \partial\overline{\Omega}_\mathrm{i} = \partial\overline{\Omega}_i \setminus \partial\Omega$.

Furthermore, we define the outer and inner boundaries

$\partial\overline{\Omega}_{i,O} = \partial\overline{\Omega}_i \cap \partial\Omega$, and $\partial\overline{\Omega}_{i,I} = \partial\overline{\Omega}_i \setminus \partial\Omega$,
$\partial\overline{\Omega}_{i,O}^+ = \partial\overline{\Omega}_i^+ \cap \partial\Omega$, and $\partial\overline{\Omega}_{i,I}^+ = \partial\overline{\Omega}_i^+ \setminus \partial\Omega$.



# Local Bilinear Forms

For any subdomain $\Omega_\star \subset \Omega$, we define $a^{\Omega_\star}, b^{\Omega_\star}: \mathcal{V}_h(\Omega_\star)\times\mathcal{V}_h(\Omega_\star) \to \mathbb{R}$ such that

$$a^{\Omega_\star}(\varphi_h, \zeta_h) := \int_{\Omega_\star} D_c \nabla\varphi_h \cdot \nabla\zeta_h \, d\Omega + \int_{\Omega_\star} \Sigma_a \, \varphi_h \, \zeta_h d\Omega + \int_{\partial\Omega_R \cap \partial\bar{\Omega}_\star} \varrho \; \varphi_h \, \zeta_h \, d\gamma \,, \qquad \forall \varphi_h, \zeta_h \in \mathcal{V}_h(\Omega_\star),$$

$$b^{\Omega_\star}(\varphi_h, \zeta_h) := \int_{\Omega_\star} \Sigma_f \, \varphi_h \, \zeta_h \, d\Omega \,, \qquad \forall \varphi_h, \zeta_h \in \mathcal{V}_h(\Omega_\star).$$

As long as the coefficients are $\Sigma_f \geq 0$, and $D_c, \Sigma_a > 0$, we have that

$$a^{\Omega_\star} \succ 0,$$
$$b^{\Omega_\star} \succcurlyeq 0.$$

# Bounding the Eigenvalue

The global eigenvalue $\mathrm{k}_{\mathrm{eff}}$ can be bounded from above and from bellow relying on the eigenvalues of problems defined only on the subdomains $\Omega_i$, and $\Omega_\mathrm{i}^+$. Specifically, we define

1) the local eigenvalue problem with **zero Neumann** boundary conditions defined as follows:
   Find the dominant eigen-couple $\phi_h^{N,i}, \mathrm{k}_{eff}^{N,i} \in \mathcal{V}_h(\Omega_\mathrm{i}) \times \mathbb{R}^+$, with $\|\phi_h\|_{\mathrm{L}^2(\Omega_\mathrm{i})} = 1$, such that

$$b^{\Omega_\mathrm{i}}\left(\phi_h^{N,i}, \zeta_h\right) = \mathrm{k}_{eff}^{N,i}\, a^{\Omega_i}\left(\phi_h^{N,i}, \zeta_h\right), \qquad \forall \zeta_h \in \mathcal{V}_h(\Omega_\mathrm{i}),$$

2) the local eigenvalue problem with **zero Dirichlet** boundary condition defined as follows:
   Find the dominant eigen-couple $\phi_h^{D,i}, \mathrm{k}_{eff}^{D,i} \in \mathcal{V}_h^0(\Omega_\mathrm{i}^+) \times \mathbb{R}^+$, with $\|\phi_h^{D,i}\|_{\mathrm{L}^2(\Omega_\mathrm{i}^+)} = 1$, such that

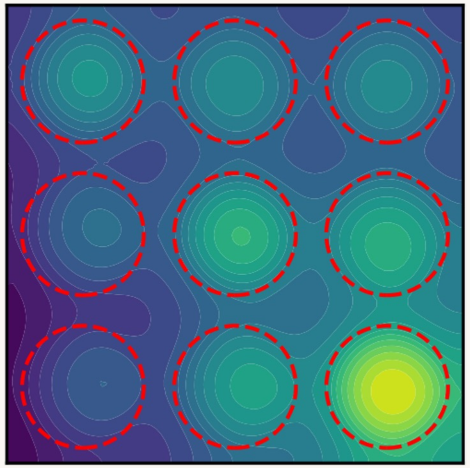
$$b^{\Omega_\mathrm{i}^+}\left(\phi_h^{D,i}, \varphi_h\right) = \mathrm{k}_{eff}^{D,i}\, a^{\Omega_\mathrm{i}^+}\left(\phi_h^{D,i}, \varphi_h\right), \qquad \forall \varphi_h \in \mathcal{V}_h^0(\Omega_\mathrm{i}^+),$$

where $\mathcal{V}_h^0(\Omega_\mathrm{i}^+) \coloneqq \{\zeta_h \in \mathcal{V}_h(\Omega_\mathrm{i}^+) \mid \mathcal{I}_{\Omega_\mathrm{i} \to \partial \bar{\Omega}_{i,I}^+} \zeta_h = 0\}$. With these, we can prove that

$$\underline{k}_{\mathrm{eff}} = \max_{0 \le i \le N_D} \mathrm{k}_{eff}^{D,i} < \mathrm{k}_{eff} \le \max_{0 \le i \le N_D} \mathrm{k}_{eff}^{N,i} = \bar{\mathrm{k}}_{eff}$$

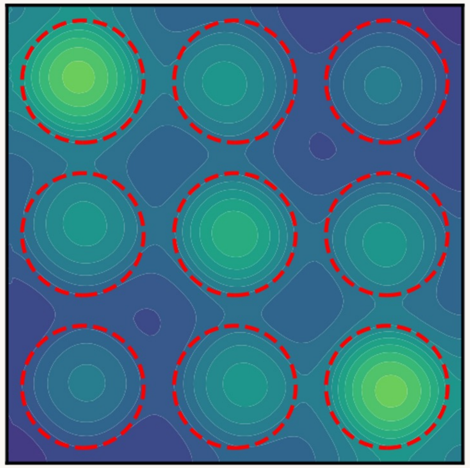
# Upper Bound

$\phi_h^{N,1}$



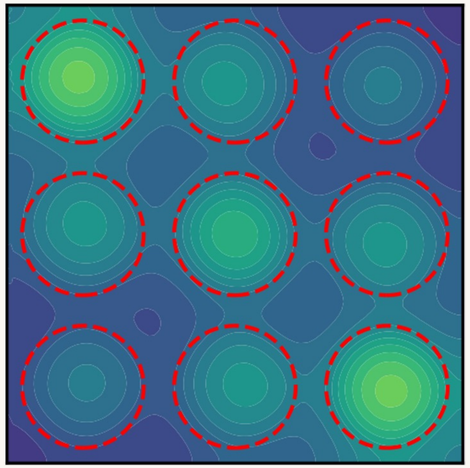
$\mathrm{k}_{eff}^{N,1} = 0.9776$

$\phi_h^{N,2}$



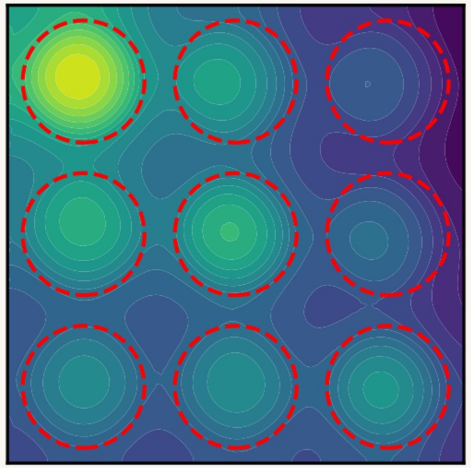
$\mathrm{k}_{eff}^{N,2} = 1.0063$

$\phi_h^{N,3}$



$\mathrm{k}_{eff}^{N,3} = 1.0063$

$\phi_h^{N,4}$

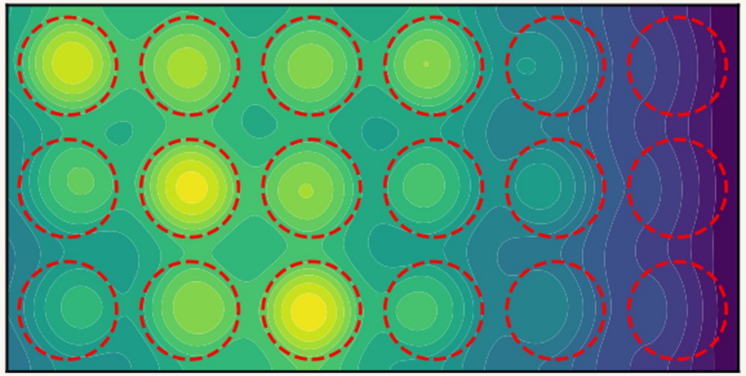


$\mathrm{k}_{eff}^{N,4} = 0.9776$

$$k_{\mathrm{eff}} = 0.9905 \leq 1.0063 = \max_{0 \leq i \leq N_D} \mathrm{k}_{eff}^{N,i}$$

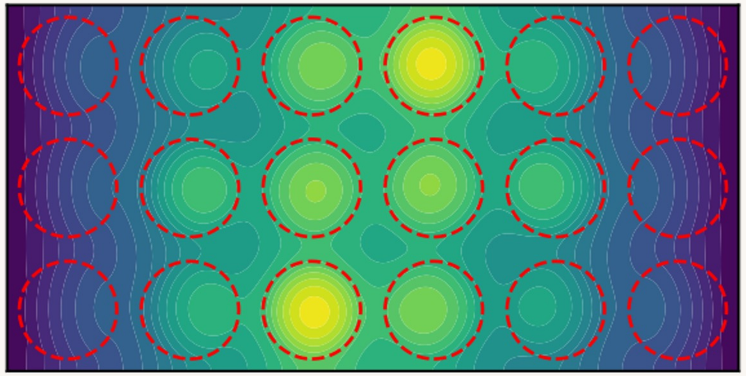
# Lower Bound

$\phi_h^{D,1}$



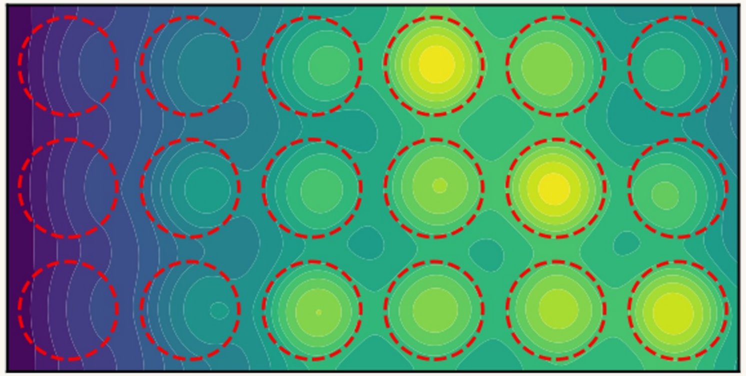
$\mathrm{k}_{eff}^{D,1} = 0.9556$

$\phi_h^{D,2}$



$\mathrm{k}_{eff}^{D,2} = 0.9207$

$\phi_h^{D,3}$



$\mathrm{k}_{eff}^{D,3} = 0.9556$

$$\max_{0 \le i \le N_D} \mathrm{k}_{eff}^{D,i} = 0.9556 \geq 0.9905 = k_{\text{eff}}$$

# ...contact me for the complete presentation

# Relevant Literature

For drafting these notes, we were inspired by the following works:

[1] *Y. Efendiev, J. Galvis, T. Y. Hou, Generalized multiscale finite element methods (GMsFEM)* Journal of computational physics, 2013

[2] P. Henning, A. Målqvist, *Localized orthogonal decomposition techniques for boundary value problems,* SIAM Journal on Scientific Computing, 2014

[3] A. Abdulle, E. Weinan, B. Engquist, E. Vanden-Eijnden, *The heterogeneous multiscale method,* Acta Numerica, 2012

[4] D. B. P. Huynh, D. J. Knezevic, A. T. Patera, *A static condensation reduced basis element method: approximation and a posteriori error estimation,* Mathematical Modelling and Numerical Analysis, 2013

[5] K. Smetana, A. T. Patera, *Optimal local approximation spaces for component-based static condensation procedures,* SIAM Journal on Scientific Computing, 2016

# THANK YOU

Texas A&M University
Department of Nuclear Engineering

francesco.silva@tamu.edu