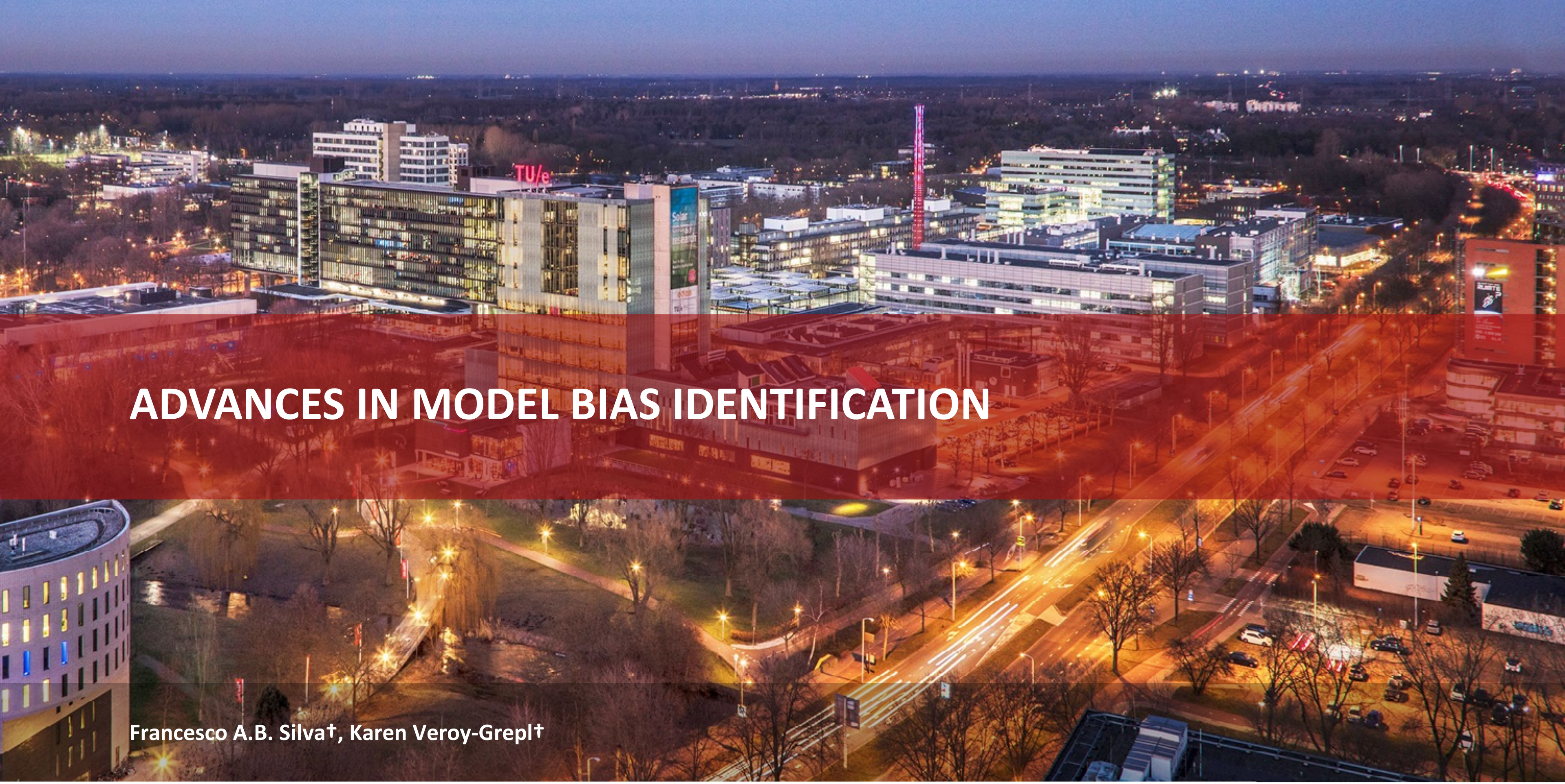

ADVANCES IN MODEL BIAS IDENTIFICATION

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ACKNOWLEDGMENTS

ERC-818473 : work supported by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation program

INTRODUCTION

STATE ESTIMATION : AN INVERSE PROBLEM

$$y = \hat{\mathbf{L}}(u_t) + \epsilon \quad \Longrightarrow \quad u_t(\mu_t, c_t)$$

DATA — STATE

STATE ESTIMATION: EMPLOYING IMPLICIT MODEL

$$\boldsymbol{y} = \hat{\mathbf{L}}(u_t) + \boldsymbol{\epsilon}$$

DATA

→

$$\boldsymbol{\beta}$$

COEFFICIENTS

→

$$u_t = \sum \beta_i \psi_i$$

STATE

STATE ESTIMATION: EMPLOYING IMPLICIT MODEL

$$y = \hat{\mathbf{L}}(u_t) + \epsilon \quad \Rightarrow \quad \boldsymbol{\beta} \quad \Rightarrow \quad u_t = \sum \beta_i \psi_i$$

DATA — COEFFICIENTS — STATE

SPACE BASE METHODS $(\boldsymbol{\beta}_i)$

- GEIM, PBDW[2]
- (Kriging)
- …

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STATE ESTIMATION: EMPLOYING IMPLICIT MODEL

$$\boldsymbol{y} = \hat{\mathbf{L}}(u_t) + \boldsymbol{\epsilon} \quad \Longrightarrow \quad \boldsymbol{\beta} \quad \Longrightarrow \quad u_t = \sum \beta_i \psi_i$$

DATA — COEFFICIENTS — STATE

SPACE BASE METHODS $(\boldsymbol{\beta}_i)$

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Requires proper approximation space and is usually more prone to noise effects because of the large number of coefficients to be estimated.

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DATA

STATE

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$$\boldsymbol{y} = \hat{\mathbf{L}}(u_t) + \epsilon \quad \Longrightarrow \quad \mu_t, c_t \quad \Longrightarrow \quad u_t(\mu_t, c_t)$$

DATA → PARAMETERS → STATE

STATE ESTIMATION: EMPLOYING EXPLICIT MODEL

$y = \hat{\mathbf{L}}(u_t) + \epsilon$ ⇒ μ_t, c_t ⇒ $u_t(\mu_t, c_t)$

DATA — PARAMETERS — STATE

MODEL BASED METHODS $(\boldsymbol{\mu_t})$

- 3D-VAR[1]
- Kalman Filters
- …

STATE ESTIMATION: EMPLOYING EXPLICIT MODEL

$y = \hat{\mathbf{L}}(u_t) + \epsilon$ $\Rightarrow$ μ_t, c_t $\Rightarrow$ $u_t(\mu_t, c_t)$

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DATA — PARAMETERS — STATE

MODEL BASED METHODS (μ_t)

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Expensive whenever the model is nonlinear and in all the cases when the parameters are not linear control variables.

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STATE ESTIMATION: AN INVERSE PROBLEM

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DATA

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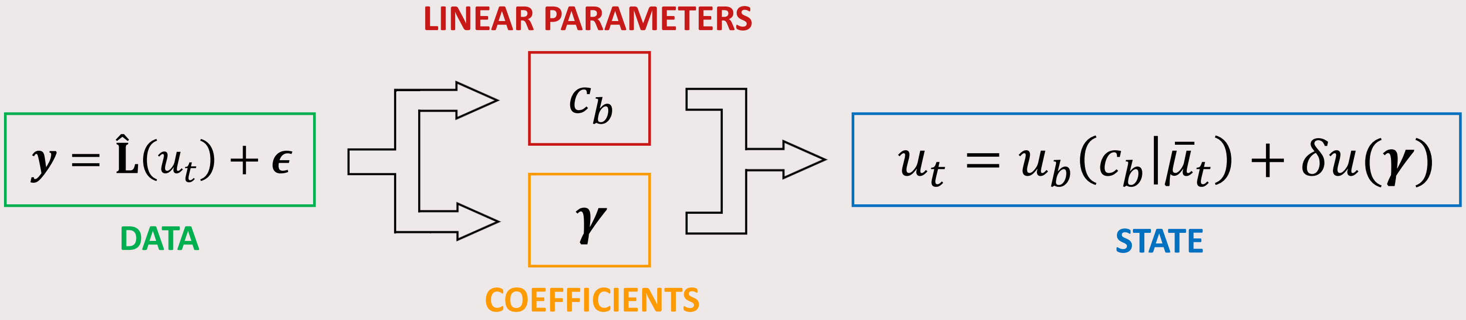
STATE ESTIMATION: EMPLOYING IMPERFECT EXPLICIT MODEL



u_b: IMPERFECT MODEL SOLUTION

c_b: IMPERFECT MODEL LINEAR PARAMETER

PRACTICAL MOTIVATION FOR IMPERFECT MODEL (COMPLEXITY)

Plenty of problems admit decent linear approximation solutions:

Heat & Mass Transfer

- Radiative HT $[\sim \mathrm{T}^4]$
- Nonlinear H&MT $[k(\mathrm{T}), \mathrm{D}(\mathrm{T}, \mathrm{c})]$

Electrodynamics

- HT Coupling $[\mu(\mathrm{T}), \varepsilon(\mathrm{T})]$
- Charge carriers diffusion $[\varepsilon(\mathrm{E}, \mathrm{c})]$

Neutron Kinetics

- Neutron Transport
- H&MT Coupling $[\chi(\mathrm{E}, \mathrm{T}, \mathrm{c})]$

Solid Mechanics

- Finite strain $[\ldots + \nabla \boldsymbol{u}^{\mathsf{T}} \cdot \nabla \boldsymbol{u}]$
- Non-Hookean Solids $[\sigma(\varepsilon)]$

EPISTEMIC MOTIVATION FOR IMPERFECT MODEL (UNCERTAINTY)

Physical Coefficients

Geometrical Parameters

Boundary Conditions

…

⇐

Even for linear problems, the direct estimation of these quantities from experimental data results in non-linear-quadratic and non-convex optimization problems.

EPISTEMIC MOTIVATION FOR IMPERFECT MODEL (UNCERTAINTY)

Physical Coefficients

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Even for linear problems, the direct estimation of these quantities from experimental data results in non-linear-quadratic and non-convex optimization problems.

We can incorporate complexity and uncertainty into a Bias Space to account for all the missing information of the Approximated Model

THE METHOD

3D-VAR : PERFECT MODEL FORMULATION

$$\boldsymbol{y} = \hat{\mathbf{L}}(u_t) + \boldsymbol{\epsilon} \quad \text{where} \quad \boldsymbol{\epsilon} \sim \mathbf{N}(\mathbf{0}, \boldsymbol{\Sigma}) \quad \text{and} \quad \mathcal{A}_\mu(u_t, \psi) = \mathcal{S}_\mu(c_t, \psi) \quad \forall \psi \in \mathcal{U}$$

DATA — NOISE — PERFECT MODEL

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DATA NOISE PERFECT MODEL

Solving the exact inverse problem means solve:

$$(u^\star, c^\star, \mu^\star) = \underset{(u,c,\mu)\in\mathcal{U}\times\mathcal{C}\times\mathcal{D}}{\operatorname{argmin}} \frac{1}{2} \|\boldsymbol{\mathcal{L}}(u) - \boldsymbol{y}\|_{\Sigma^{-1}}^2$$

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Non-linear and non-convex constrained optimization. u is nonlinear both in c, due to the nonlinearity of the model, and in μ.

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DATA

NOISE

APPROXIMATED LINEAR MODEL

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APPROXIMATED LINEAR MODEL

with $u_t - u_b = \delta u$

MODEL BIAS

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We can approximate the inverse problem :

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δu accounts for the model bias and for the error in μ. It's assumed to be a small correction.

ASYMPTOTIC BEHAVIOURS

Case of $\boldsymbol{\lambda} \to \mathbf{0}$ $(\delta u^\star, \delta c^\star \text{ free})$

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$$\mathcal{V}_{\overline{\mu}} := \left\{ v \in \mathcal{U} \quad \text{s.t.} \;\; \mathrm{A}_{\overline{\mu}}(v, \psi) = \mathrm{S}_{\overline{\mu}}(c, \psi) \right\}$$

$$\forall \psi \in \mathcal{U}, \forall c \in \mathcal{C}$$

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⇩

Reconstructed state is to least square approximation in the space $\mathcal{B} \cup \mathcal{V}_{\bar{\mu}}$. It may be exact.

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$$\text{s.t.} \quad \mathrm{A}_{\bar{\mu}}(v,\psi) = \mathrm{S}_{\bar{\mu}}(c,\psi) \quad \forall \psi \in \mathcal{U}$$

Reconstruction reduced 3D-VAR, based on a bias model. Less coefficients to be estimated.

3D-VAR : BIASED MODEL FORMULATION

$\boldsymbol{y} = \hat{\mathbf{L}}(u_t) + \boldsymbol{\epsilon}$ where $\boldsymbol{\epsilon} \sim \mathbf{N}(\mathbf{0}, \boldsymbol{\Sigma})$ and $\mathrm{A}_{\overline{\mu}}(u_b, \psi) = \mathrm{S}_{\overline{\mu}}(c_b, \psi) \quad \forall \psi \in \mathcal{U}$

DATA

NOISE

APPROXIMATED LINEAR MODEL

with $u_t - u_b = \delta u$

MODEL BIAS

We can approximate the inverse problem :

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HOW TO SOLVE?

SOLUTION THROUGH LAGRANGIAN OPTIMIZATION

$$\mathcal{L}(u, \delta u, \delta c, p) = \frac{1}{2}\left\|\hat{\mathbf{L}}(u) - \boldsymbol{y}\right\|_{\Sigma^{-1}}^{2} + \frac{\lambda}{2}\|\delta u\|_{\mathcal{U}}^{2} + \frac{\gamma}{2}\|\delta c\|_{\mathcal{C}}^{2} + \mathrm{A}_{\overline{\mu}}(u - \delta u, p) - \mathrm{S}_{\overline{\mu}}(\bar{c} + \delta c, p)$$

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A first order optimality condition can be easily derived:

$$\begin{cases} \\ \\ \mathrm{A}_{\overline{\mu}}(u, \phi) - \mathrm{A}_{\overline{\mu}}(\delta u, \phi) - \mathrm{S}_{\overline{\mu}}(\delta c, \phi) = \mathrm{S}_{\overline{\mu}}(\bar{c}, \phi) \qquad \forall \phi \in \mathcal{U} \end{cases}$$

Approximated Model

SOLUTION THROUGH LAGRANGIAN OPTIMIZATION

$$\mathcal{L}(u, \delta u, \delta c, p) = \frac{1}{2}\left\|\hat{\mathbf{L}}(u) - \boldsymbol{y}\right\|_{\Sigma^{-1}}^{2} + \frac{\lambda}{2}\|\delta u\|_{\mathcal{U}}^{2} + \frac{\gamma}{2}\|\delta c\|_{\mathcal{C}}^{2} + \mathrm{A}_{\overline{\mu}}(u - \delta u, p) - \mathrm{S}_{\overline{\mu}}(\bar{c} + \delta c, p)$$

A first order optimality condition can be easily derived:

$$\begin{cases} \hat{\mathbf{L}}(\psi)^{\mathrm{T}}\Sigma^{-1}\hat{\mathbf{L}}(u) + \mathrm{A}_{\overline{\mu}}(\psi, p) = \hat{\mathbf{L}}(\psi)^{\mathrm{T}}\Sigma^{-1}\boldsymbol{y} & \forall \psi \in \mathcal{U} \\ \lambda(\varphi, \delta u)_{\mathcal{U}} - \mathrm{A}_{\overline{\mu}}(\varphi, p) = 0 & \forall \varphi \in \mathcal{B} \\ \gamma(\eta, \delta u)_{\mathcal{C}} - \mathrm{S}_{\overline{\mu}}(\eta, p) = 0 & \forall \eta \in \mathcal{C} \\ \mathrm{A}_{\overline{\mu}}(u, \phi) - \mathrm{A}_{\overline{\mu}}(\delta u, \phi) - \mathrm{S}_{\overline{\mu}}(\delta c, \phi) = \mathrm{S}_{\overline{\mu}}(\bar{c}, \phi) & \forall \phi \in \mathcal{U} \end{cases}$$

Adjoint Equations (first three equations)

Approximated Model (last equation)

SADDLE POINT PROBLEM – IS IT WELL-POSED?

$$\begin{cases} \hat{\mathbf{L}}(\psi)^{\mathrm{T}}\Sigma^{-1}\hat{\mathbf{L}}(u) + \mathrm{A}_{\overline{\mu}}(\psi, p) & = \hat{\mathbf{L}}(\psi)^{\mathrm{T}}\Sigma^{-1}\boldsymbol{y} & \forall\psi \in \mathcal{U} \\ \lambda(\varphi, \delta u)_{\mathcal{U}} - \mathrm{A}_{\overline{\mu}}(\varphi, p) & = 0 & \forall\varphi \in \mathcal{B} \\ \gamma(\eta, \delta u)_{\mathcal{C}} - \mathrm{S}_{\overline{\mu}}(\eta, p) & = 0 & \forall\eta \in \mathcal{C} \\ \mathrm{A}_{\overline{\mu}}(u, \phi) - \mathrm{A}_{\overline{\mu}}(\delta u, \phi) - \mathrm{S}_{\overline{\mu}}(\delta c, \phi) & = \mathrm{S}_{\overline{\mu}}(\bar{c}, \phi) & \forall\phi \in \mathcal{U} \end{cases}$$

Well-posedness can be proved proven assuming that A_{μ}, S_{μ}, $\hat{\mathbf{L}}$ are bounded

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$$
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\end{cases}
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Well-posedness can be proved proven assuming that A_{μ}, S_{μ}, $\hat{\mathbf{L}}$ are bounded, and that:

$$
\alpha_{\mathrm{L}} \coloneqq \inf_{\mu\in\mathcal{D}} \inf_{u\in\mathcal{U}} \frac{\mathrm{A}_{\mu}(u\, u)}{\|u\|_{\mathcal{U}}^{2}} > 0
$$

Coercivity

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Well-posedness can be proved proven assuming that A_{μ}, S_{μ}, $\hat{\mathbf{L}}$ are bounded, and that:

$$\alpha_{\mathrm{L}} \coloneqq \inf_{\mu\in\mathcal{D}} \inf_{u\in\mathcal{U}} \frac{\mathrm{A}_{\mu}(u\, u)}{\|u\|_{\mathcal{U}}^2} > 0 \qquad \eta_{\mathrm{L}} \coloneqq \inf_{\mu\in\mathcal{D}} \inf_{v\in\mathcal{V}_{\mu}} \frac{\|v\|_{\mathcal{U}}^2}{\|c\|_{\mathcal{C}}^2} > 0$$

Coercivity   Controllability

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Controllability

$$\beta_{\mathrm{L}} \coloneqq \inf_{\mu\in\mathcal{D}} \inf_{v\in\mathcal{V}_{\mu}} \frac{\left\|\hat{\mathbf{L}}(v)\right\|_{\Sigma^{-1}}^2}{\|v\|_{\mathcal{U}}^2} > 0$$

Observability

SPACES CONSTRUCTION AND DISCRETIZATION

We need 3 functional spaces to set up the problem:

$$\mathcal{U} \longrightarrow \mathcal{U} \approx \mathcal{U}_h \text{ or } \mathcal{U}_{\mathrm{R}} = \mathrm{MOR}\{\mathcal{V}\}$$

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NB : If we choose for $\mathcal{B}$ the space spanned by the Rietz representations of the elements of the linear functional $\hat{\mathbf{L}}$, the method coincides with the APBDW [3]

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$\mathcal{C}_{\mathrm{R}}$

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$$\mathcal{V}_{\mathrm{R}} \approx \mathrm{span}\{v\}$$

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Where:

(a) Constr. the Biased Model RB Space

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$$\mathcal{V}_{\mathrm{R}} \approx \mathrm{span}\{v\}$$
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(c) Constr. the Model RB Space

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$$\mathcal{V}_{\mathrm{R}} \approx \mathrm{span}\{v\}$$
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$$\mathcal{C}_{\mathrm{R}} \xrightarrow{(a)} \mathcal{V}_{\mathrm{R}} \xrightarrow{(b)} \mathcal{B}_{\mathrm{R}} \xrightarrow{(c)} \mathcal{W}_{\mathrm{R}} \xrightarrow{(d)} \hat{\mathbf{L}}$$

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(d) Constr. the Measurements op. $\hat{\mathbf{L}}$ [4]

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(b) Constr. the Adjoint RB Space

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(c) Constr. the Model RB Space

(d) Constr. the Measurements op. $\hat{\mathbf{L}}$ [4]

(b) Constr. the Adjoint RB Space

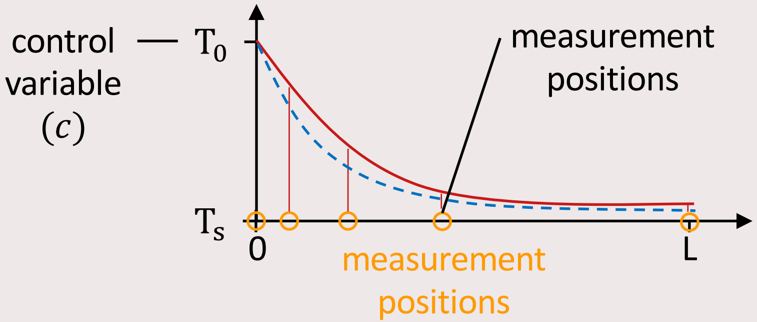
(f) Constr. the Global RB Space $\mathcal{U}_\mathrm{R} = \mathrm{span}\{\mathcal{V}_\mathrm{R}, \mathcal{Y}_\mathrm{R}\}$

A PRATICAL EXAMPLE

PRACTICAL EXAMPLE - 1D RADIATIVE HEAT TRANSFER

Exact Model

$$-kA\frac{d^2T}{dx^2} = -P\varepsilon\sigma(T^4 - T_s^4)$$
$$+ \text{b.c.}$$



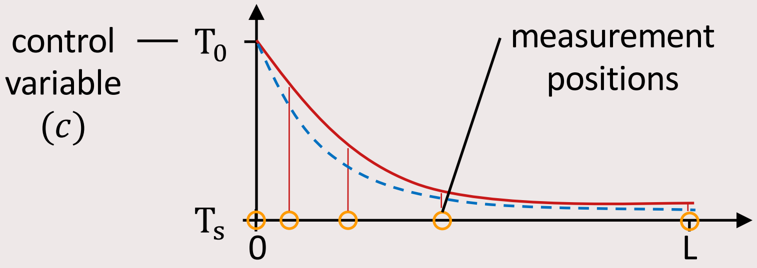
PRACTICAL EXAMPLE - 1D RADIATIVE HEAT TRANSFER

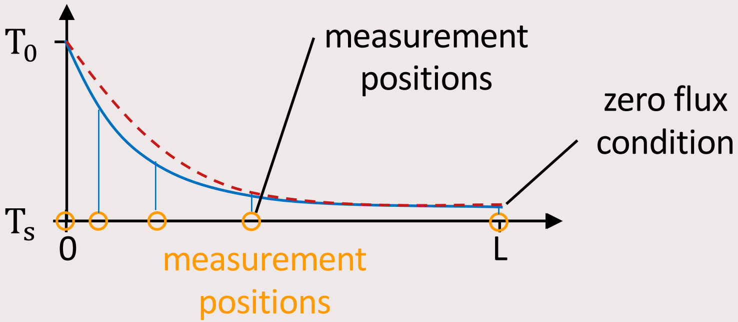
Exact Model

$$-kA\frac{d^2T}{dx^2} = -P\varepsilon\sigma(T^4 - T_s^4)$$
$$+ \text{b.c.}$$

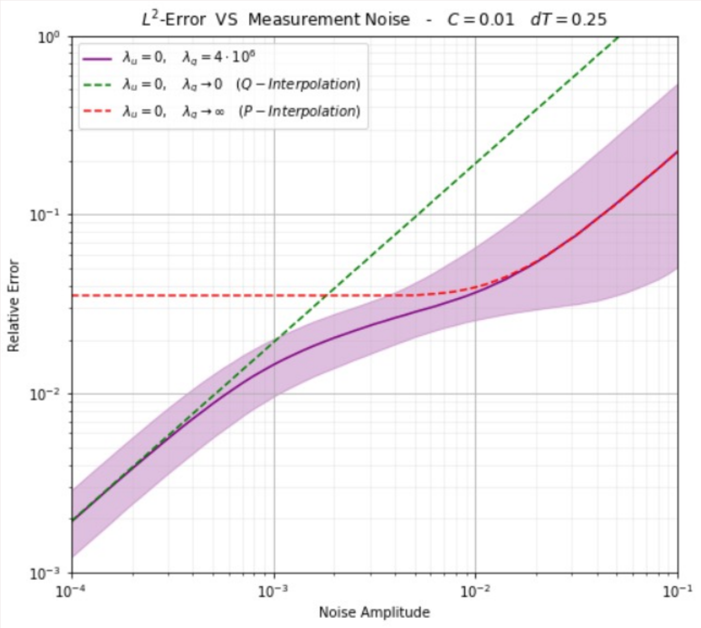
Biased Model

$$-kA\frac{d^2T}{dx^2} = -4P\varepsilon\sigma T_s^3(T - T_s)$$
$$+ \text{b.c.}$$





PRACTICAL EXAMPLE - NOISE VS RELATIVE ERROR



error

noise

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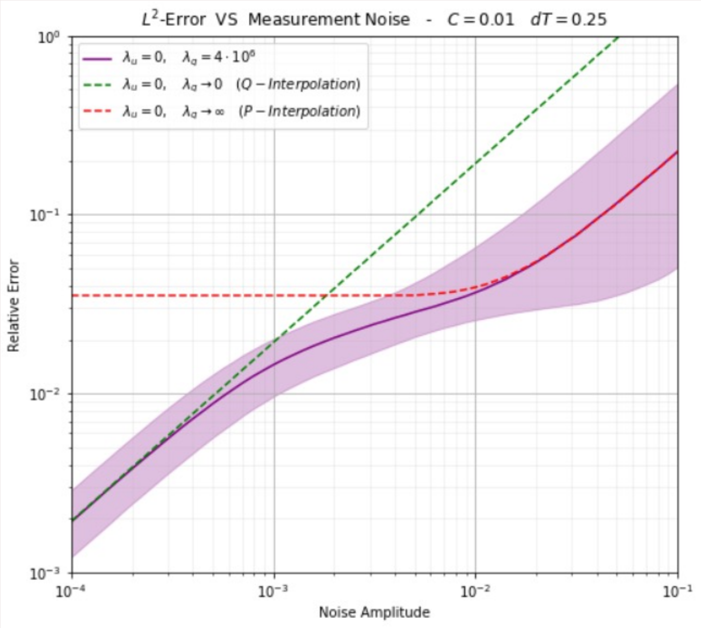

For $\lambda \to 0$, the method reduces to a least square estimation in the Bias Space $(\mathcal{B})$. It's optimal in noise free conditions.

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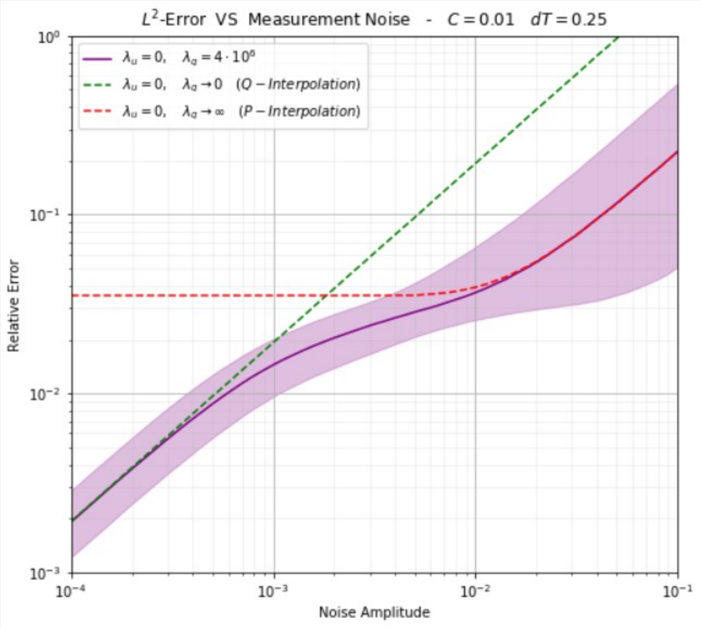

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For $\lambda \to \infty$, the method reduces to the 3D-VAR employing the Biased Model. In noise free conditions the error coincide with the Bias energy.

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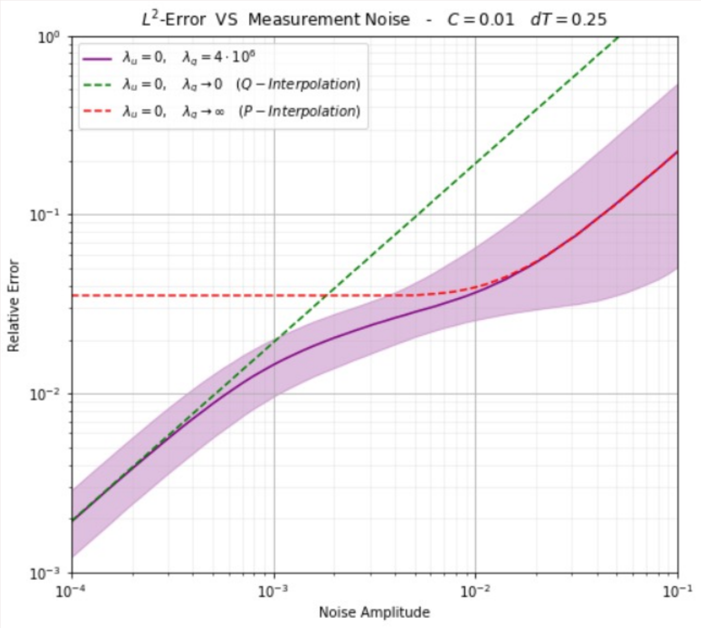

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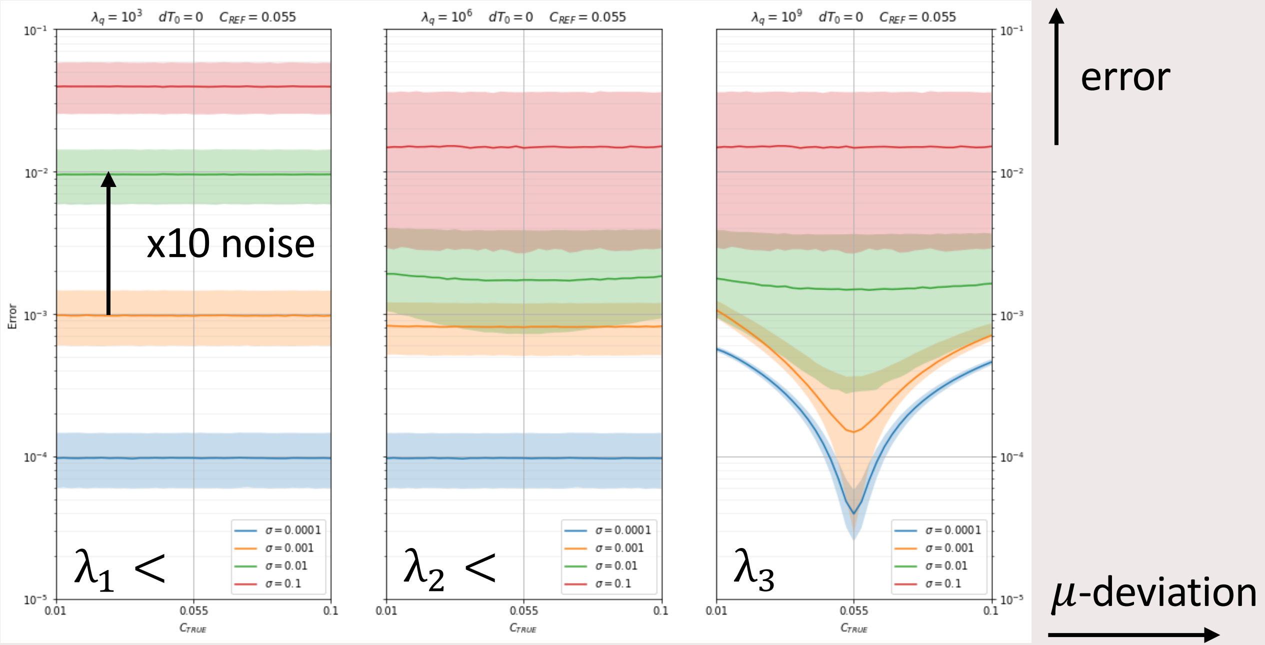
For $\lambda \to \infty$, the method reduces to the 3D-VAR employing the Biased Model. In noise free conditions the error coincide with the Bias energy.

It exists an optimal λ, which optimally bridges the two behaviors. The transition occurs when noise and bias are comparable.

PRACTICAL EXAMPLE - MODEL BIAS AND NOISE EFFECTS



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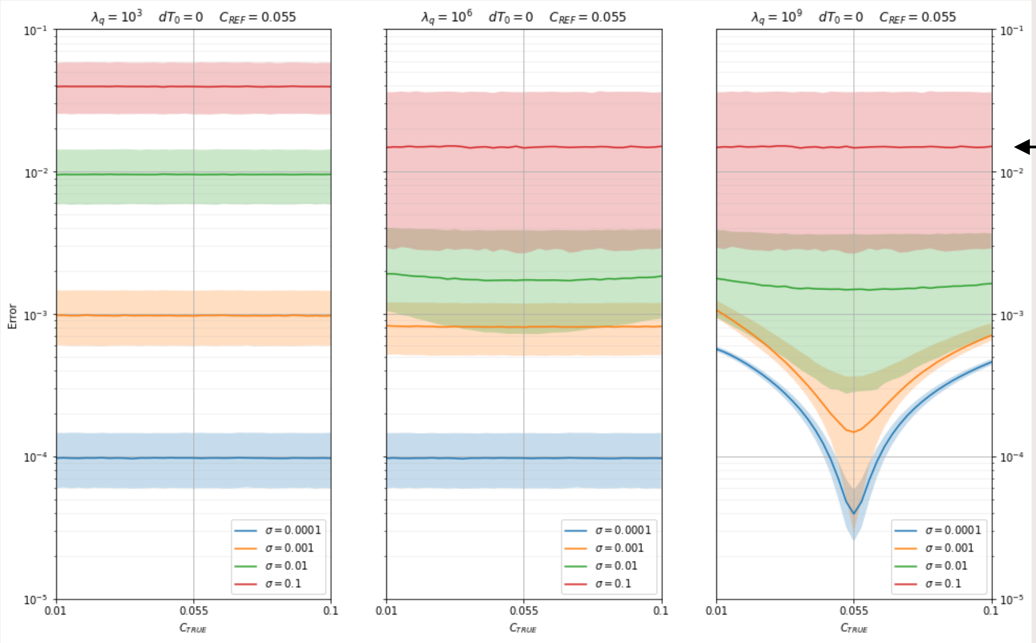

A larger λ mitigates error in presence of large noise. The smaller number of coefficients estimated is dominant.

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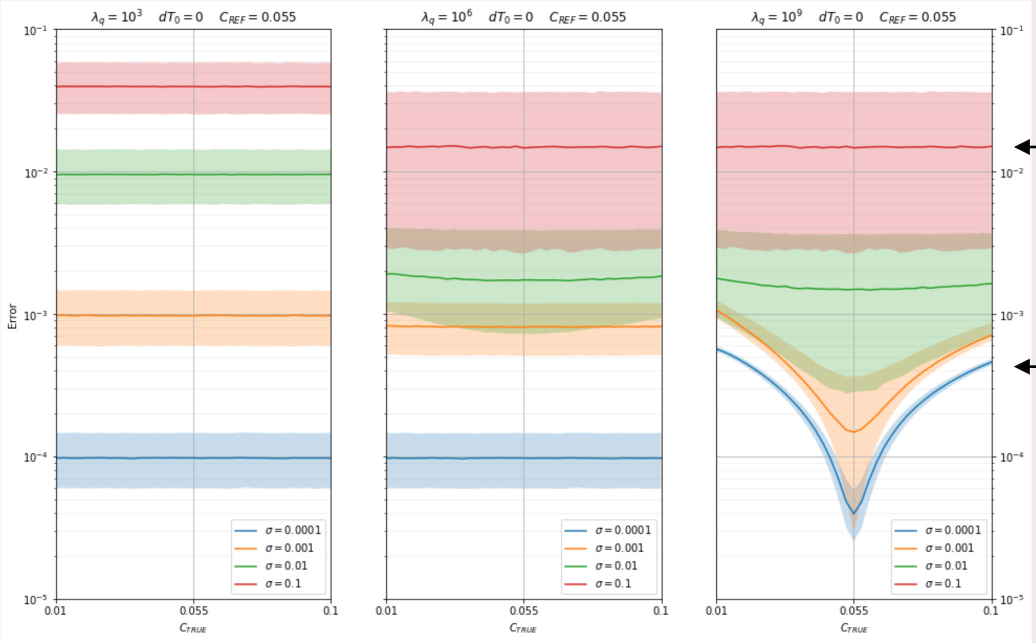

A larger λ mitigates error in presence of large noise. The smaller number of coefficients estimated is dominant.

In weak noise conditions, a large λ may correspond to poorer reconstructions. That's because the bias effects are not mitigated.

CONCLUSIONS

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- We’ve developed a new method bridging two approaches to state estimation
- We’ve proven the well-posedness of the method and tested over a toy problem
- We observed that a prior knowledge on the magnitude of the bias is still required

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FUTURE DEVELOPMENTS

- We’re developing bounds for the variance of the reconstruction error
- We’re studying more complex problems in more than 1D
- We’re thinking to extend the method to time dependent problems

RELATED LITERATURE

[1] Maday, Y., Patera, A.T., Penn, J.D., Yano, M. : A parameterized background data-weak approach to variational data assimilation: formulation, analysis, and application to acoustics (2014)

[2] Aretz-Nellesen, N., Grepl, M.A., Veroy, K. : 3D-VAR for Parametrized Partial Differential Equations: A Certified Reduced Basis Approach (2019)

[3] Maday, Y., Taddei,T. : Adaptive PBDW Approach to State Estimation: Noisy Observations; UserDefined Update Spaces (2019)

[4] Binev, P., Cohen, A., Mula, O., Nichols, J. : Greedy algorithms for optimal measurements selection in state estimation using reduced models (2018)

THANK YOU FOR YOU ATTENTION!

TIME FOR QUESTIONS!