A multi-fidelity ensemble Kalman filter with adaptive reduced-order models

Francesco A. B. Silva, Cecilia Pagliantini, Karen Veroy

Department of Mathematics and Computer Science, CASA Group

e-mail : f.a.b.silva@tue.nl

EINDHOVEN UNIVERSITY OF TECHNOLOGY

COLLABORATORS

Karen Veroy : Centre for Analysis, Scientific Computing and Applications, TU\e

Cecilia Pagliantini : Department of Mathematics, University of Pisa

COLLABORATORS

Karen Veroy : Centre for Analysis, Scientific Computing and Applications, TU\e

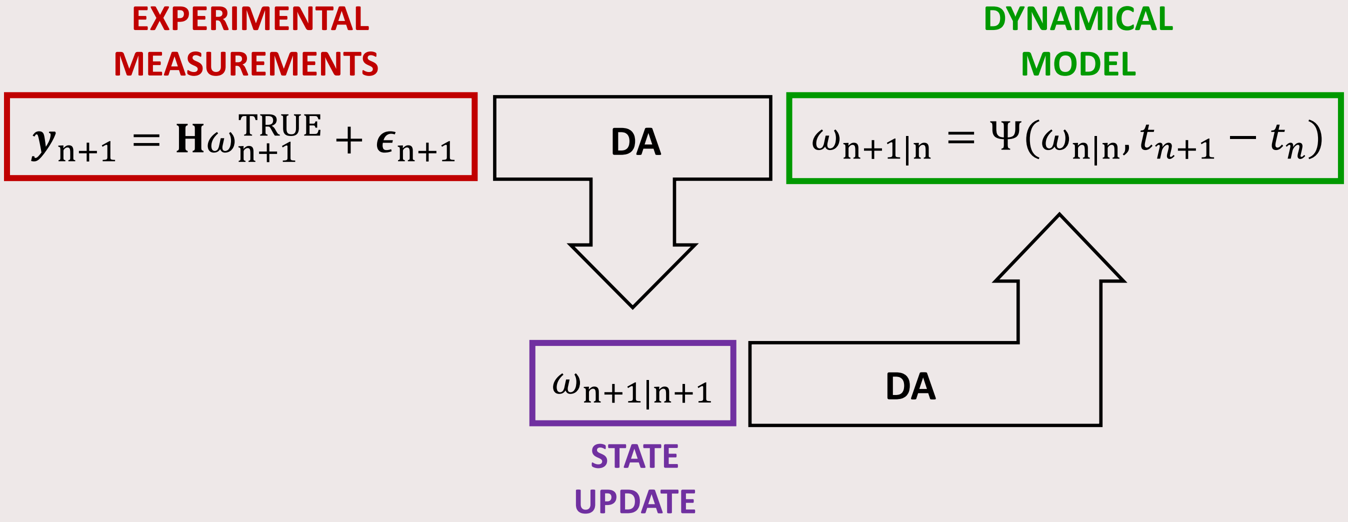
Cecilia Pagliantini : Department of Mathematics, University of Pisa

ACKNOWLEDGMENTS

ERC-818473 : work supported by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation program

DATA ASSIMILATION

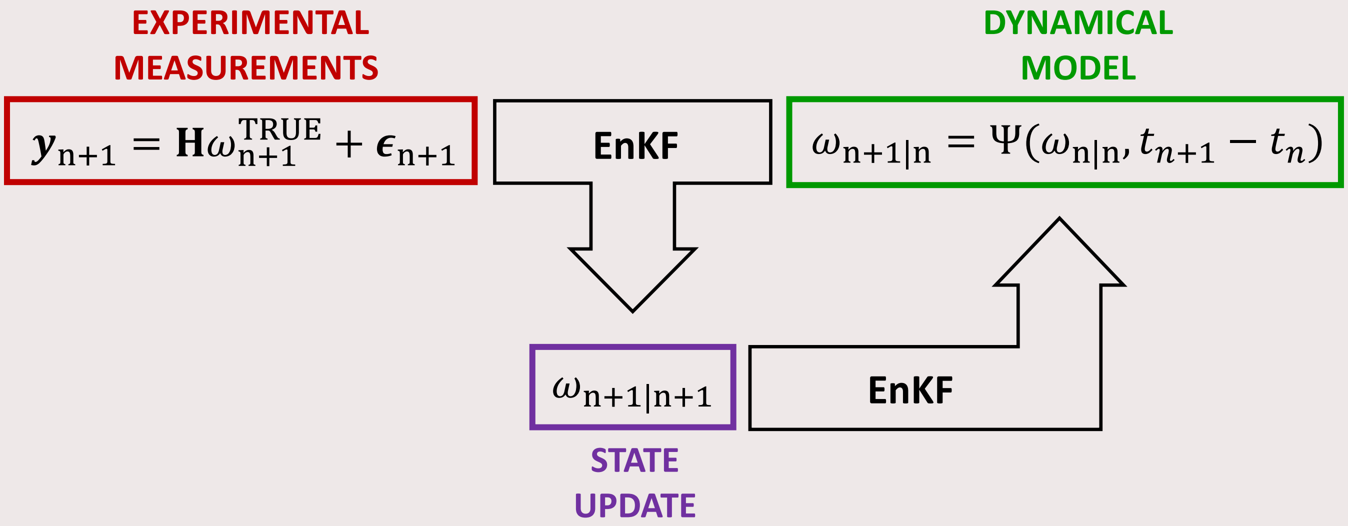
$$y_n \in \mathcal{Y}$$
$$\omega_{k|n} \in \mathcal{V}$$



[Eve03]

DATA ASSIMILATION

G. Evensen. "The ensemble Kalman filter: Theoretical formulation and practical implementation". (2003)



THE ENSEMBLE KALMAN FILTER

PREDICT : $$\omega^{\mathrm{i}}_{\mathrm{n+1|n}} = \Psi(\omega^{\mathrm{i}}_{\mathrm{n|n}}, t_{n+1} - t_n)$$

ESTIMATE : $$\mathbf{C}_{\mathrm{n+1|n}} \approx \mathrm{cov}\left(\omega^{\mathrm{i}}_{\mathrm{n+1|n}}, \omega^{\mathrm{i}}_{\mathrm{n+1|n}}\right)$$

ANALYSE : $$\omega^{\mathrm{i}}_{\mathrm{n+1|n+1}} = \omega^{\mathrm{i}}_{\mathrm{n+1|n}} + \mathbf{K}_{\mathrm{n}} \left(\boldsymbol{y}_{\mathrm{n+1}} - \mathbf{H}\omega^{\mathrm{i}}_{\mathrm{n+1|n}}\right)$$

THE ENSEMBLE KALMAN FILTER

PREDICT : $\omega^{\mathrm{i}}_{\mathrm{n+1|n}} = \Psi(\omega^{\mathrm{i}}_{\mathrm{n|n}}, t_{n+1} - t_n)$

ESTIMATE : $\mathbf{C}_{\mathrm{n+1|n}} \approx \mathrm{cov}\left(\omega^{\mathrm{i}}_{\mathrm{n+1|n}}, \omega^{\mathrm{i}}_{\mathrm{n+1|n}}\right)$

ANALYSE : $\omega^{\mathrm{i}}_{\mathrm{n+1|n+1}} = \omega^{\mathrm{i}}_{\mathrm{n+1|n}} + \mathbf{K}_{\mathrm{n}}\left(\boldsymbol{y}_{\mathrm{n+1}} - \mathbf{H}\omega^{\mathrm{i}}_{\mathrm{n+1|n}}\right)$

employing the Kalman gain

$$\mathbf{K}_{\mathrm{n}} = \mathbf{C}_{\mathrm{n+1|n}}\mathbf{H}^*(\mathbf{H}\mathbf{C}_{\mathrm{n+1|n}}\mathbf{H}^* + \boldsymbol{\Sigma})^{-1}$$

● $\in \mathcal{V}$

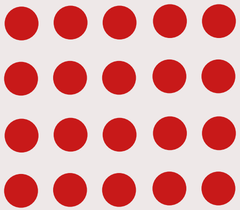
THE ENSEMBLE KALMAN FILTER

PREDICT : $\omega^{\mathrm{i}}_{\mathrm{n+1|n}} = \Psi(\omega^{\mathrm{i}}_{\mathrm{n|n}}, t_{n+1} - t_n)$ ⟵ **EXPENSIVE!**

ESTIMATE : $\mathbf{C}_{\mathrm{n+1|n}} \approx \mathrm{cov}\left(\omega^{\mathrm{i}}_{\mathrm{n+1|n}}, \omega^{\mathrm{i}}_{\mathrm{n+1|n}}\right)$

ANALYSE : $\omega^{\mathrm{i}}_{\mathrm{n+1|n+1}} = \omega^{\mathrm{i}}_{\mathrm{n+1|n}} + \mathbf{K}_{\mathrm{n}}\left(\boldsymbol{y}_{\mathrm{n+1}} - \mathbf{H}\omega^{\mathrm{i}}_{\mathrm{n+1|n}}\right)$

(i)



employing the Kalman gain

$$\mathbf{K}_{\mathrm{n}} = \mathbf{C}_{\mathrm{n+1|n}}\mathbf{H}^*(\mathbf{H}\mathbf{C}_{\mathrm{n+1|n}}\mathbf{H}^* + \boldsymbol{\Sigma})^{-1}$$

$\bullet \in \mathcal{V}$

$\cdot \in \mathcal{V}_\varepsilon \subseteq \mathcal{V}$

THE ENSEMBLE KALMAN FILTER

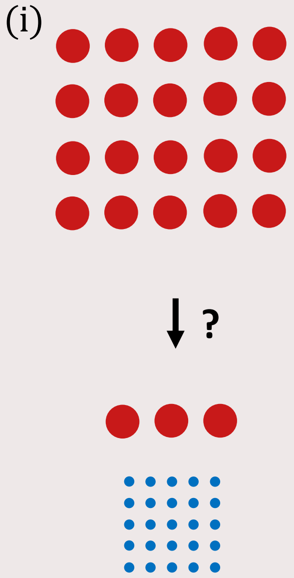
PREDICT : $\omega^{\mathrm{i}}_{\mathrm{n+1|n}} = \Psi(\omega^{\mathrm{i}}_{\mathrm{n|n}}, t_{n+1} - t_n)$

ESTIMATE : $\mathbf{C}_{\mathrm{n+1|n}} \approx \mathrm{cov}\left(\omega^{\mathrm{i}}_{\mathrm{n+1|n}}, \omega^{\mathrm{i}}_{\mathrm{n+1|n}}\right)$

ANALYSE : $\omega^{\mathrm{i}}_{\mathrm{n+1|n+1}} = \omega^{\mathrm{i}}_{\mathrm{n+1|n}} + \mathbf{K}_{\mathrm{n}}\,(\boldsymbol{y}_{\mathrm{n+1}} - \mathbf{H}\omega^{\mathrm{i}}_{\mathrm{n+1|n}})$

employing the Kalman gain

$$\mathbf{K}_{\mathrm{n}} = \mathbf{C}_{\mathrm{n+1|n}}\mathbf{H}^*(\mathbf{H}\mathbf{C}_{\mathrm{n+1|n}}\mathbf{H}^* + \boldsymbol{\Sigma})^{-1}$$



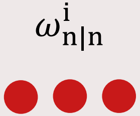
THE MULTI-ACCURACY ENSEMBLE KALMAN FILTERS

$\omega^{i}_{n|n}$

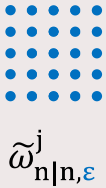
Principal Ensemble

THE MULTI-ACCURACY ENSEMBLE KALMAN FILTERS

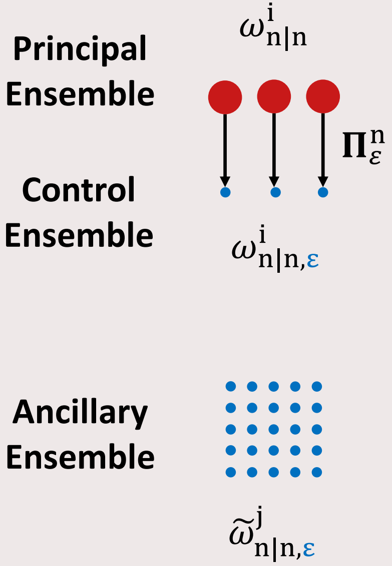
Principal Ensemble



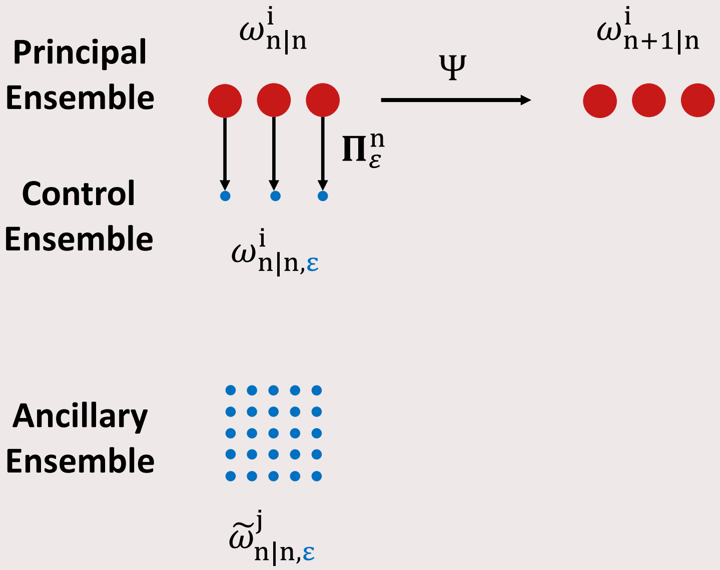
Ancillary Ensemble



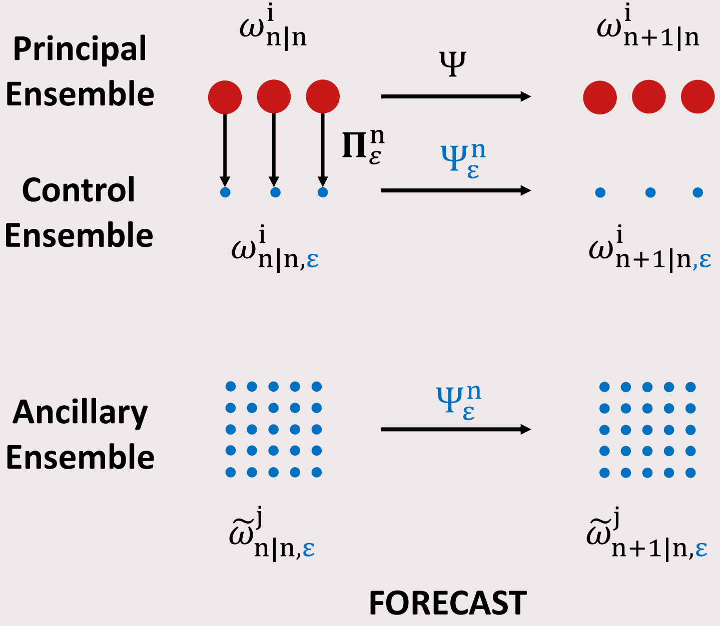
THE MULTI-ACCURACY ENSEMBLE KALMAN FILTERS



THE MULTI-ACCURACY ENSEMBLE KALMAN FILTERS



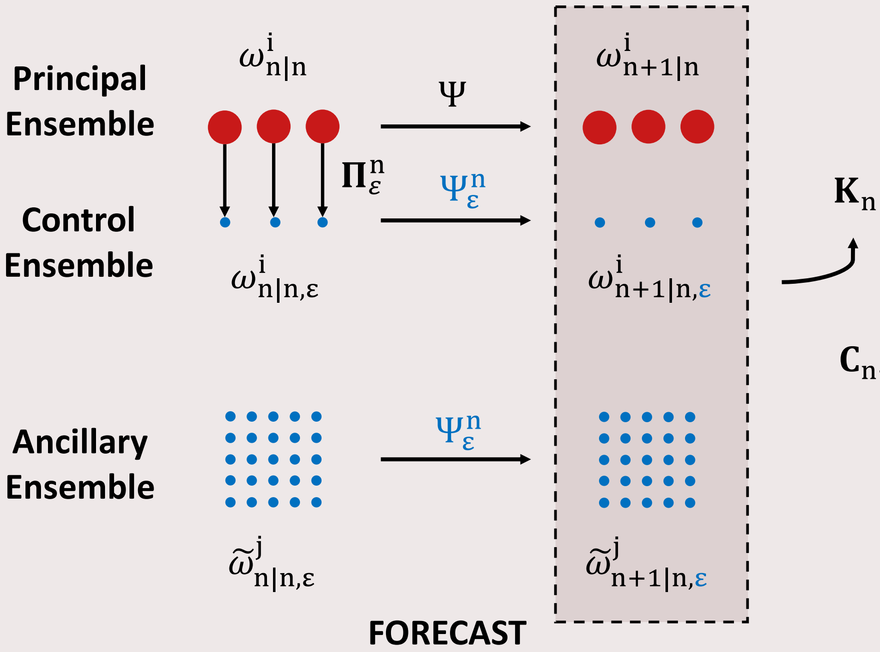
THE MULTI-ACCURACY ENSEMBLE KALMAN FILTERS



THE MULTI-FIDELITY ENSEMBLE KALMAN FILTER

[PMS21]

A. Popov, et all. "A multifidelity ensemble Kalman filter with reduced order control variates." (2021)



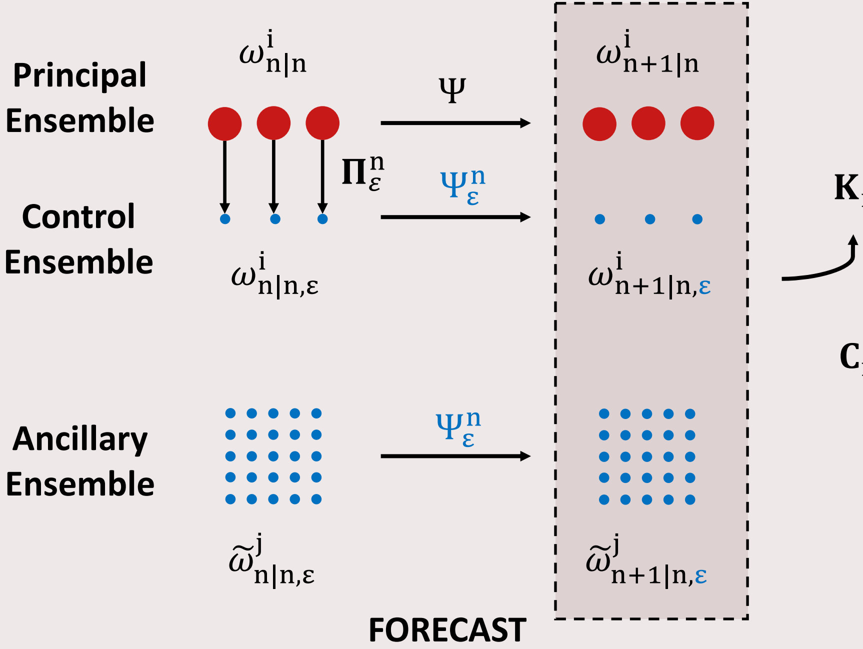
$$\mathbf{K}_n = \mathbf{C}_{n+1|n}\mathbf{H}^*(\mathbf{H}\mathbf{C}_{n+1|n}\mathbf{H}^* + \boldsymbol{\Sigma})^{-1}$$

$$\begin{aligned}
\mathbf{C}_{n+1|n} \approx\ & \operatorname{cov}\left(\omega^{i}_{n+1|n}, \omega^{i}_{n+1|n}\right) \\
& - \tfrac{1}{2}\operatorname{cov}\left(\omega^{i}_{n+1|n,\varepsilon}, \omega^{i}_{n+1|n}\right) \\
& - \tfrac{1}{2}\operatorname{cov}\left(\omega^{i}_{n+1|n}, \omega^{i}_{n+1|n,\varepsilon}\right) \\
& + \tfrac{1}{4}\operatorname{cov}\left(\omega^{i}_{n+1|n,\varepsilon}, \omega^{i}_{n+1|n,\varepsilon}\right) \\
& + \tfrac{1}{4}\operatorname{cov}\left(\widetilde{\omega}^{j}_{n+1|n,\varepsilon}, \widetilde{\omega}^{j}_{n+1|n,\varepsilon}\right)
\end{aligned}$$

THE MULTI-LEVEL ENSEMBLE KALMAN FILTER

[Che21]

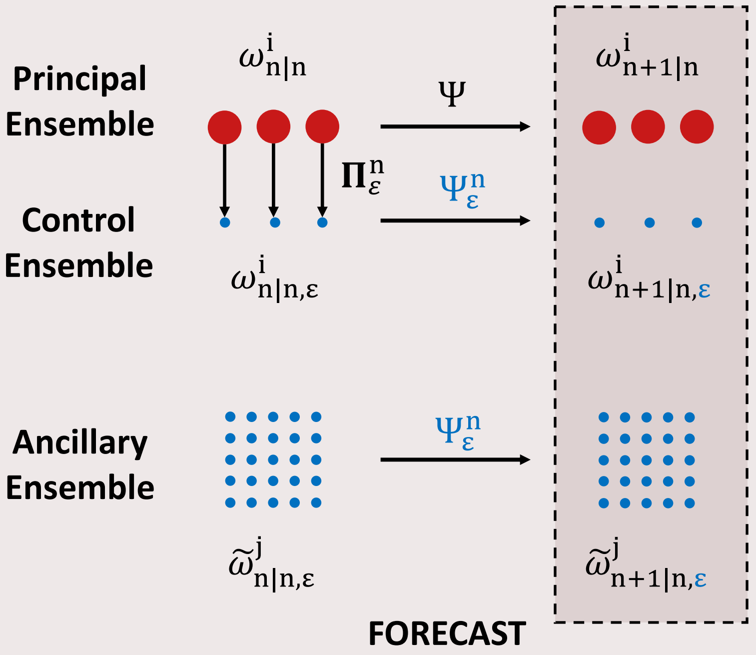
A. Chernov, et all. ”Multilevel ensemble Kalman filter for spatio-temporal processes." (2021)



$$\mathbf{K}_{n} = \mathbf{C}_{n+1|n}\mathbf{H}^{*}(\mathbf{H}\mathbf{C}_{n+1|n}\mathbf{H}^{*} + \boldsymbol{\Sigma})^{-1}$$

$$\begin{aligned}\mathbf{C}_{n+1|n} \approx\ & \mathrm{cov}\left(\omega^{i}_{n+1|n}, \omega^{i}_{n+1|n}\right) \\ & - \mathrm{cov}\left(\omega^{i}_{n+1|n,\varepsilon}, \omega^{i}_{n+1|n,\varepsilon}\right) \\ & + \mathrm{cov}\left(\tilde{\omega}^{j}_{n+1|n,\varepsilon}, \tilde{\omega}^{j}_{n+1|n,\varepsilon}\right)\end{aligned}$$

THE MULTI-LEVEL ENSEMBLE KALMAN FILTER

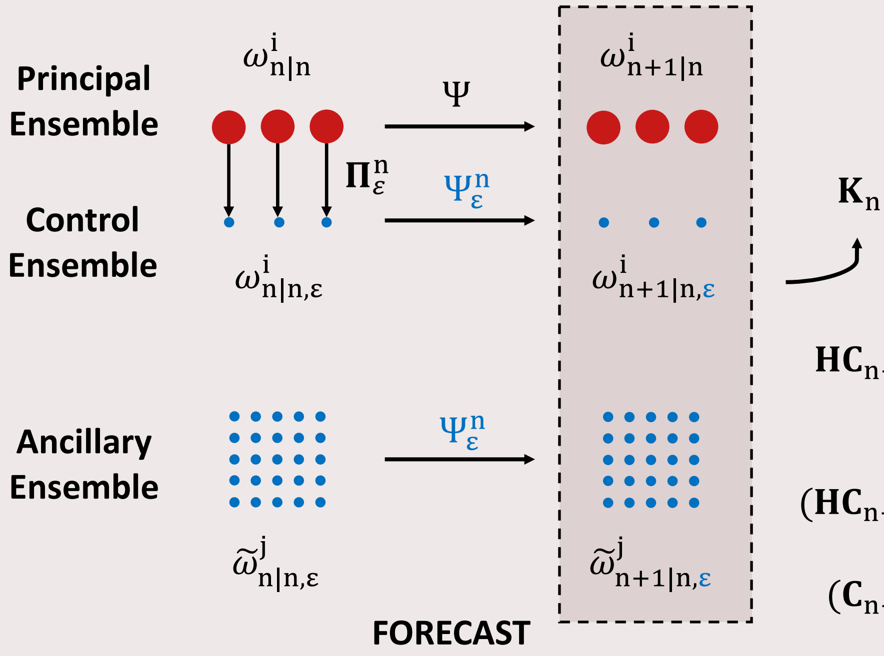


$$\mathbf{K}_n = \mathbf{C}_{n+1|n}\mathbf{H}^*(\mathbf{H}\mathbf{C}_{n+1|n}\mathbf{H}^* + \boldsymbol{\Sigma})^{-1}$$

$$\mathbf{C}_{n+1|n} \nsucc 0$$

Francesco A.B. Silva

THE MULTI-LEVEL ENSEMBLE KALMAN FILTER



$$\mathbf{K}_n = \mathbf{C}_{n+1|n}\mathbf{H}^*(\mathbf{H}\mathbf{C}_{n+1|n}\mathbf{H}^* + \mathbf{\Sigma})^{-1}$$

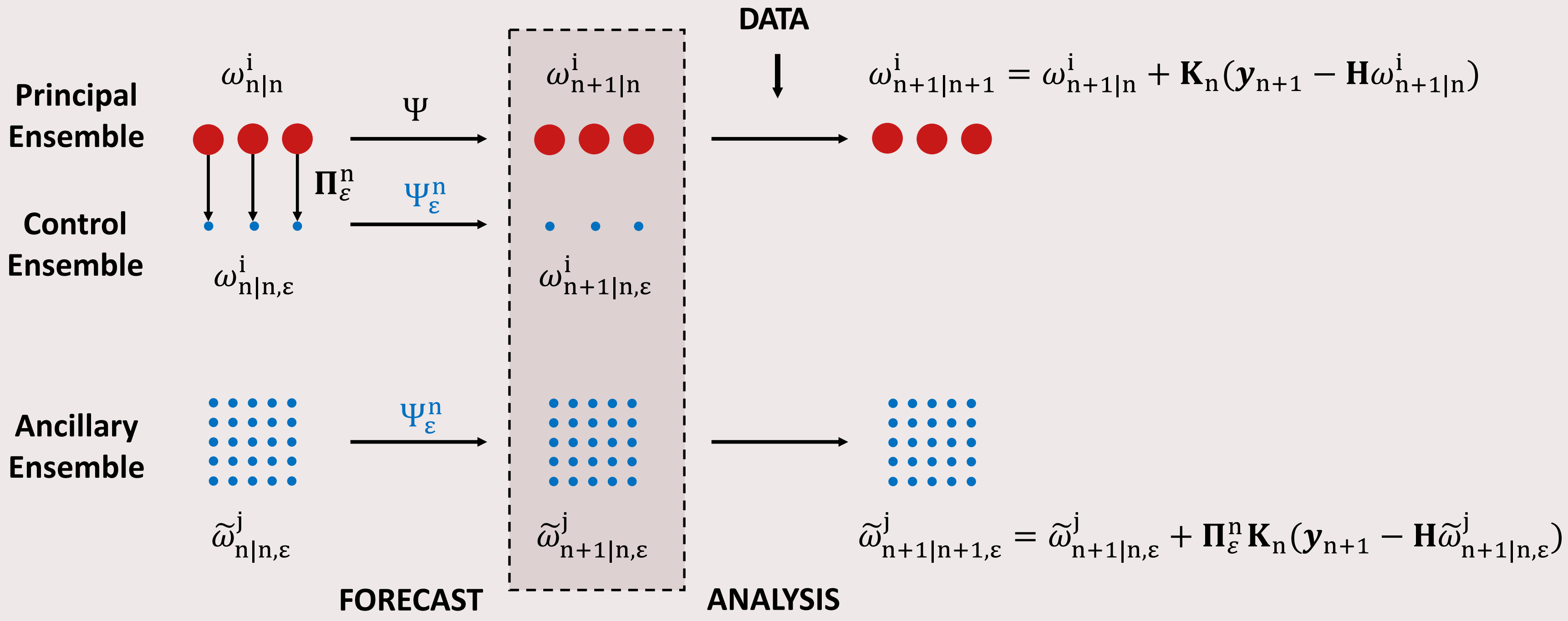
$$\mathbf{H}\mathbf{C}_{n+1|n}\mathbf{H}^* p_i = \lambda_i p_i$$

$$\downarrow$$

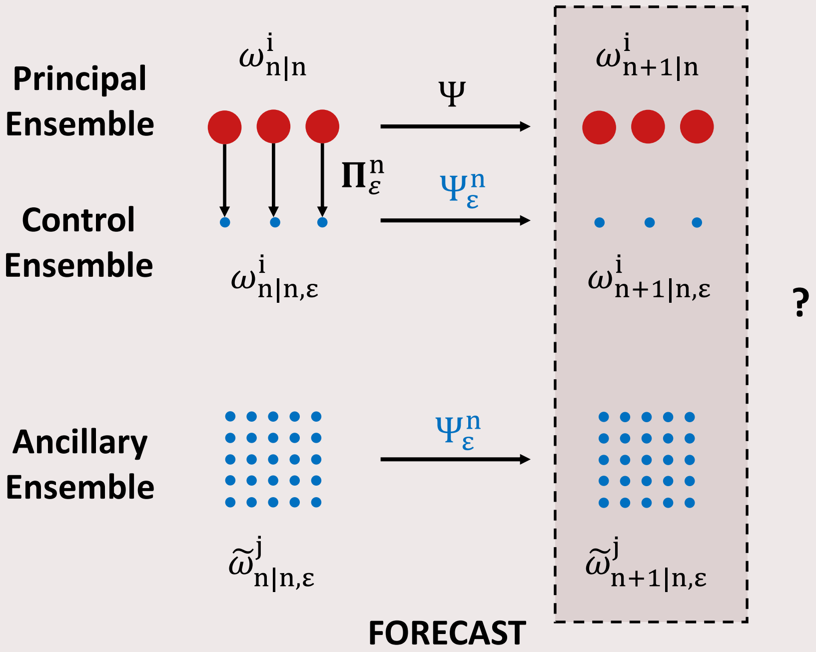
$$(\mathbf{H}\mathbf{C}_{n+1|n}\mathbf{H}^*)^+ = \sum_{i=1,\, \lambda_i>0}^{N_y} (\mathbf{H}\mathbf{C}_{n+1|n}\mathbf{H}^* p_i) p_i^*$$

$$(\mathbf{C}_{n+1|n}\mathbf{H}^*)^+ = \sum_{i=1,\, \lambda_i>0}^{N_y} (\mathbf{C}_{n+1|n}\mathbf{H}^* p_i) p_i^*$$

THE MULTI-ACCURACY ENSEMBLE KALMAN FILTERS

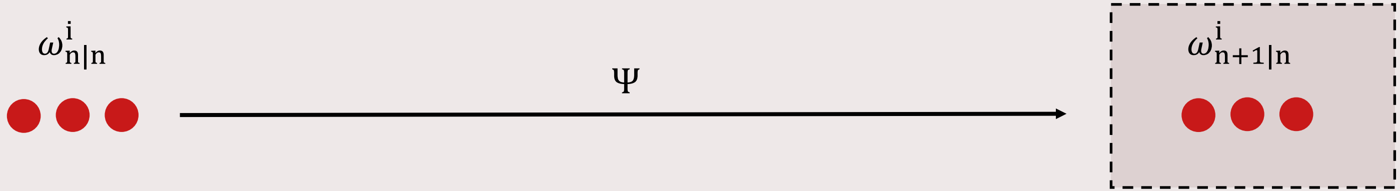


THE MULTI-ACCURACY ENSEMBLE KALMAN FILTERS



ADAPTIVE TRAINING : AN INFLATION—DEFLATION APPROACH

Principal Ensemble

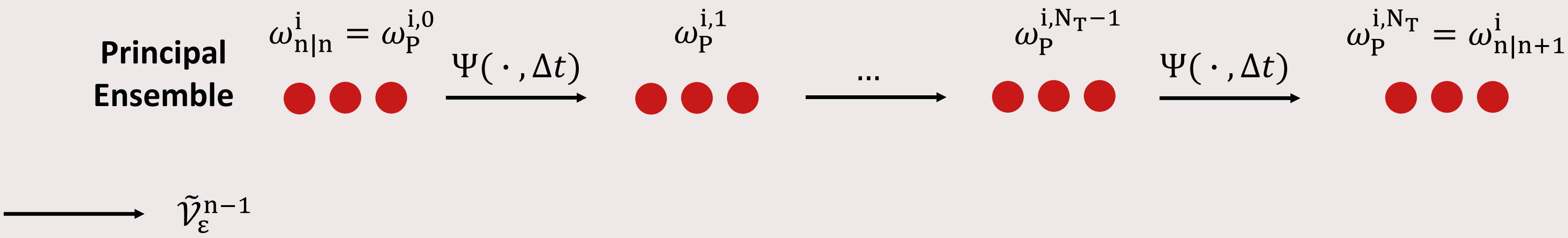


ADAPTIVE TRAINING : AN INFLATION—DEFLATION APPROACH

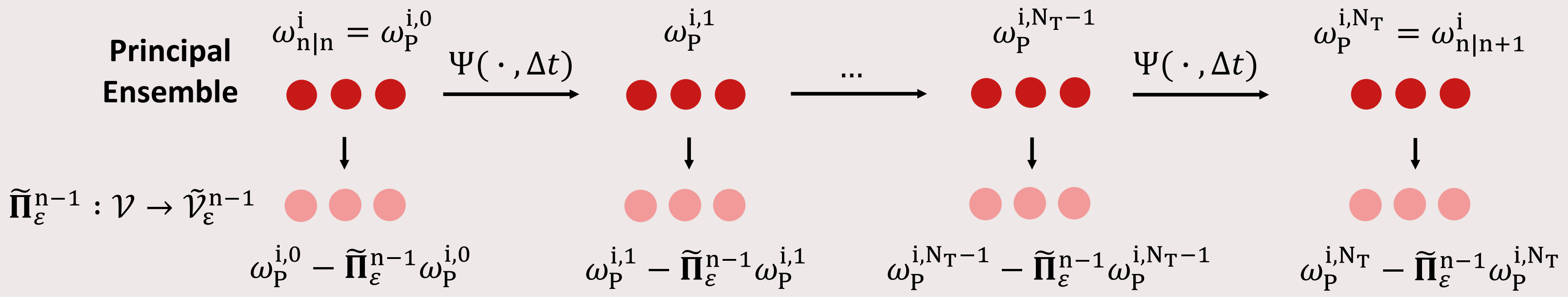
Principal Ensemble

$\omega^{\mathrm{i}}_{\mathrm{n|n}} = \omega^{\mathrm{i},0}_{\mathrm{P}} \xrightarrow{\Psi(\,\cdot\,,\Delta t)} \omega^{\mathrm{i},1}_{\mathrm{P}} \xrightarrow{\ldots} \omega^{\mathrm{i},\mathrm{N_T}-1}_{\mathrm{P}} \xrightarrow{\Psi(\,\cdot\,,\Delta t)} \omega^{\mathrm{i},\mathrm{N_T}}_{\mathrm{P}} = \omega^{\mathrm{i}}_{\mathrm{n|n+1}}$

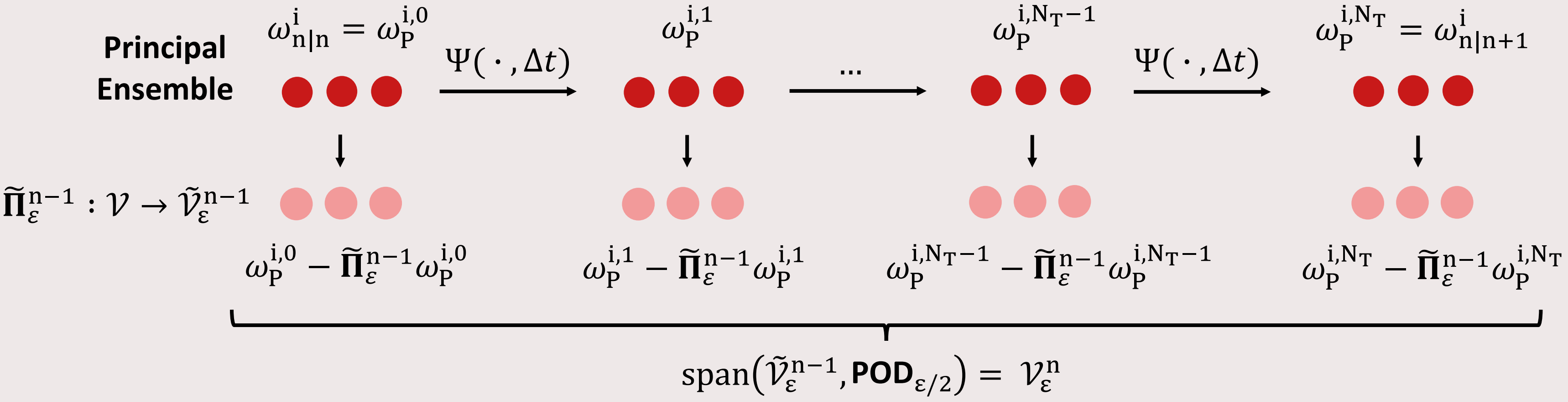
ADAPTIVE TRAINING : AN INFLATION—DEFLATION APPROACH



ADAPTIVE TRAINING : AN INFLATION—DEFLATION APPROACH

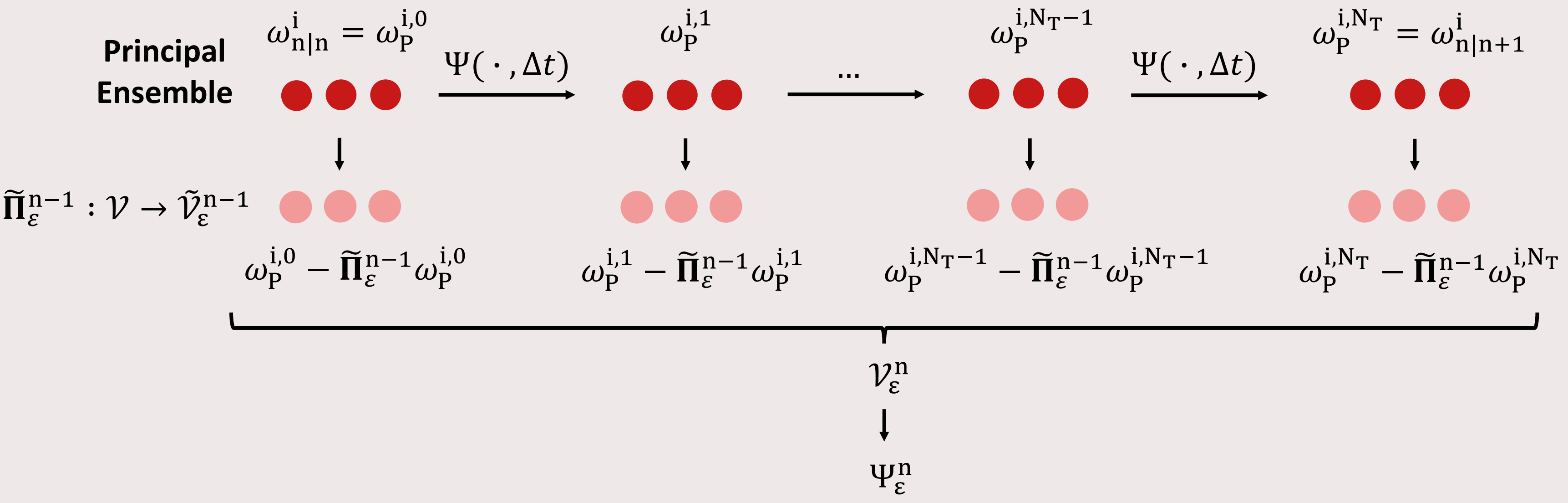


ADAPTIVE TRAINING : AN INFLATION—DEFLATION APPROACH

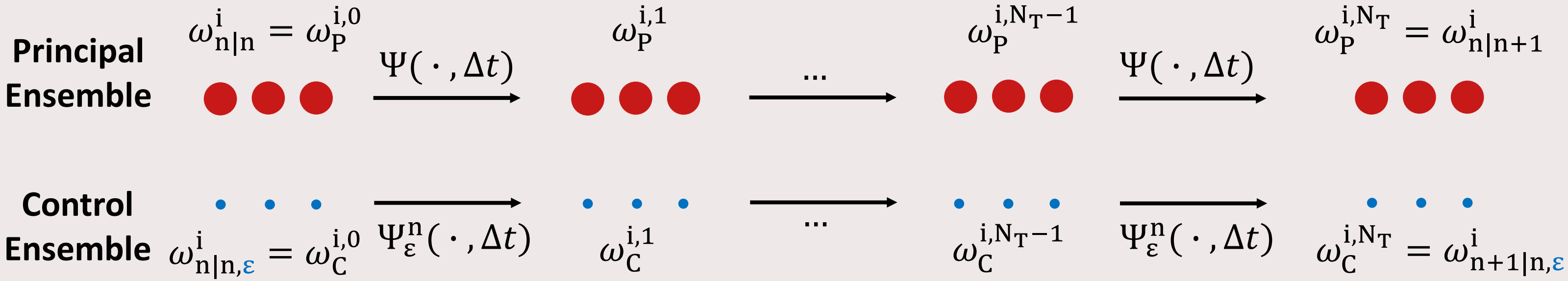


INFLATION

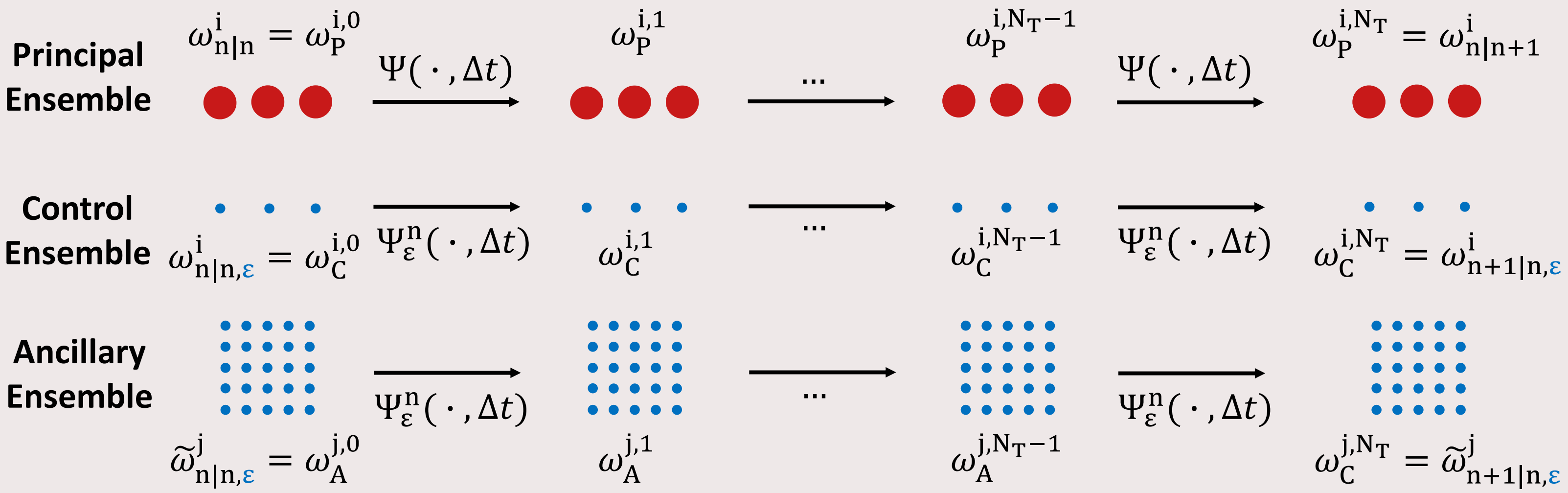
ADAPTIVE TRAINING : AN INFLATION—DEFLATION APPROACH



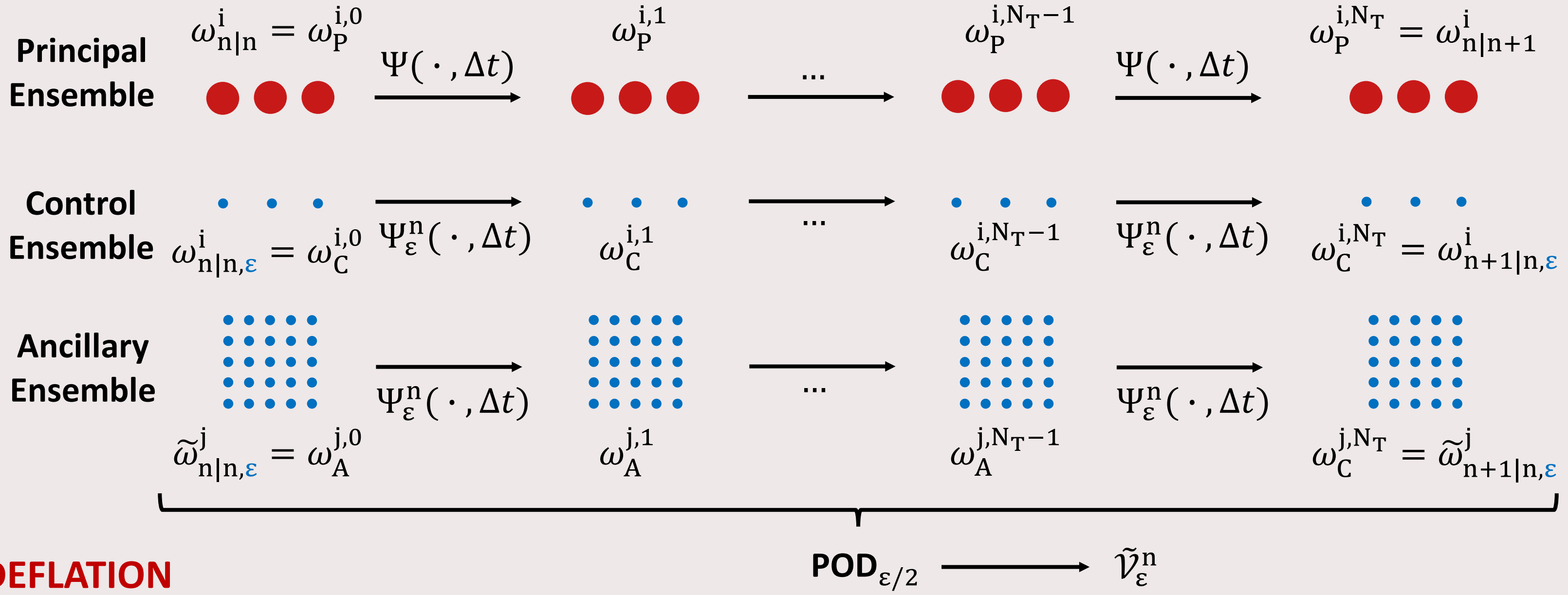
ADAPTIVE TRAINING : AN INFLATION—DEFLATION APPROACH



ADAPTIVE TRAINING : AN INFLATION—DEFLATION APPROACH

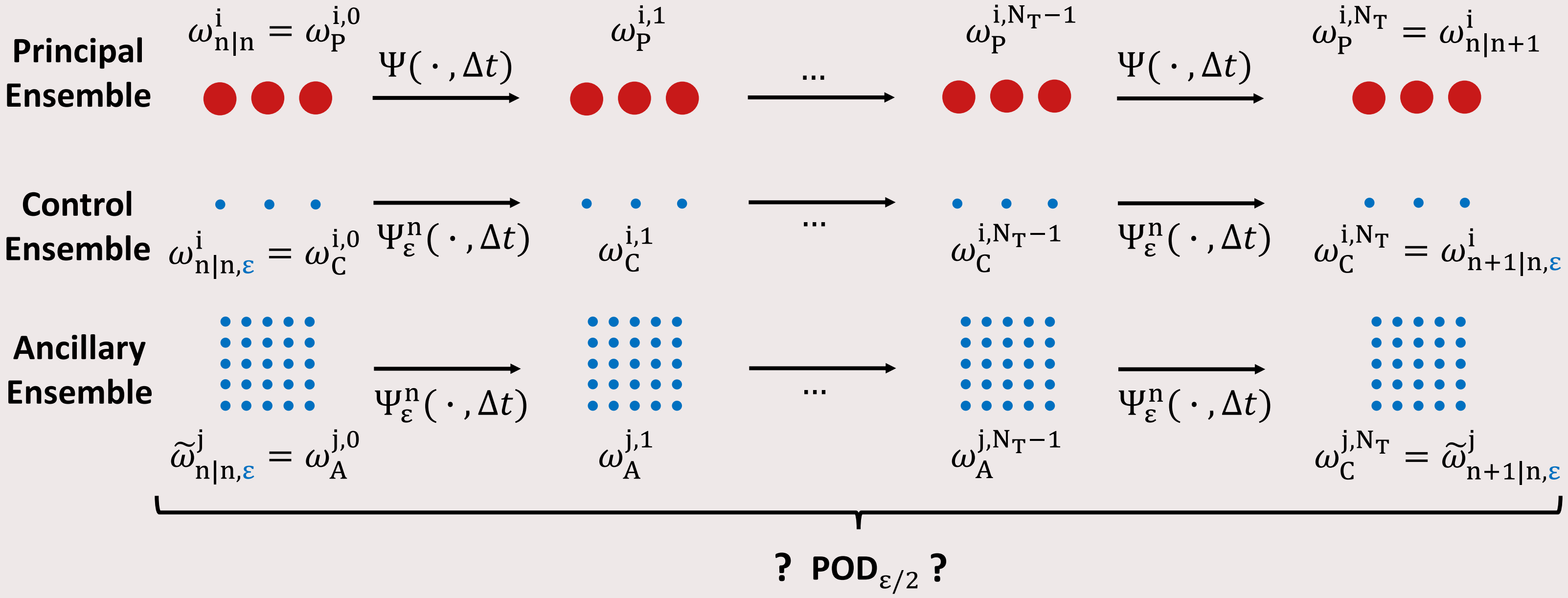


ADAPTIVE TRAINING : AN INFLATION—DEFLATION APPROACH



DEFLATION

ADAPTIVE TRAINING : AN INFLATION—DEFLATION APPROACH



ADAPTIVE TRAINING : POD TOLERANCE

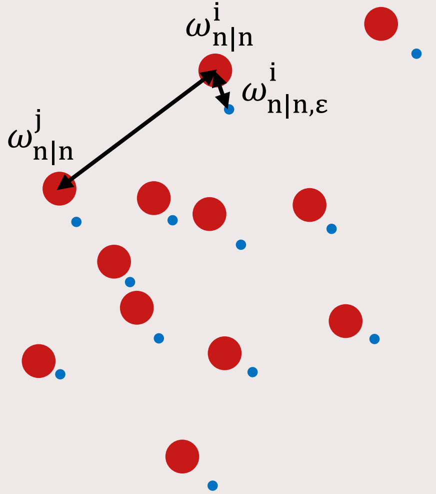
the reduced order modeling algorithm selects a space $\tilde{\mathcal{V}}_{\varepsilon}^{\mathrm{n}} \subset \mathcal{V}$ such that the average projection error is bellow a prescribed tolerance

$$\mathrm{err} = \mathrm{avg}\left(\left\|\omega_{\mathrm{n|n}}^{\mathrm{i}} - \omega_{\mathrm{n|n},\varepsilon}^{\mathrm{i}}\right\|_{\mathcal{V}}^{2}\right) < \mathrm{tol}$$

a suitable choice for this tolerance is given by

$$\mathrm{tol} = \varepsilon_r \, \mathrm{avg}\left(\left\|\omega_{\mathrm{n|n}}^{\mathrm{i}} - \omega_{\mathrm{n|n}}^{\mathrm{j}}\right\|_{\mathcal{V}}^{2}\right) = 2\varepsilon_r \mathrm{Trace}\left(\mathrm{cov}\left(\omega_{\mathrm{n|n}}^{\mathrm{i}}, \omega_{\mathrm{n|n}}^{\mathrm{i}}\right), \;\cdot\; \right)_{\mathcal{V}}$$

the trace of the covariance can be estimated using a multi-level or multi-fidelity method, consistently with the data assimilation algorithm



Francesco A.B. Silva

QUASI-GEOSTROPHIC EQUATIONS

[MWW20]

C. Mou, Z. Wang, D. Wells, X. Xie and T. Iliescu. "Reduced order models for the quasi-geostrophic equations: A brief survey." (2020)

find $\omega = \omega(x,y,t)$, $\psi = \psi(x,y,t)$ such that

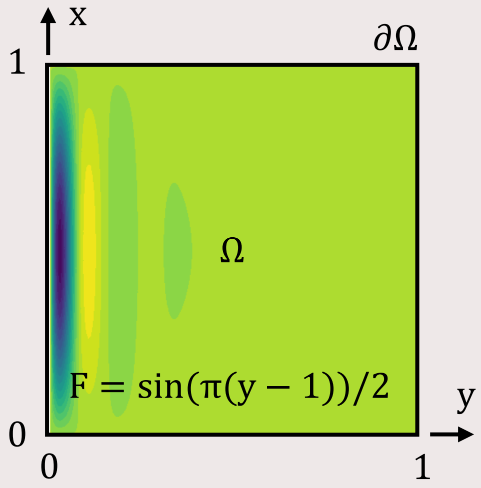
$$\partial_t \omega = \mathrm{Ro}\, J(\omega,\psi) + \partial_x \psi + \frac{\mathrm{Ro}}{\mathrm{Re}} \Delta\omega + F, \qquad \Delta\psi + \omega = 0 \longleftarrow J(\omega,\psi) = \partial_x\psi\partial_y\omega - \partial_x\psi\partial_y\omega$$

given the boundary and initial conditions

$$\omega(x,y,t) = 0, \qquad (x,y) \in \partial\Omega$$
$$\psi(x,y,t) = 0, \qquad (x,y) \in \partial\Omega$$
$$\omega(x,y,0) = \omega_0, \qquad (x,y) \in \Omega$$

and

$$\partial_x\psi_0 + \frac{\mathrm{Ro}}{\mathrm{Re}}\Delta\omega_0 + F = 0, \quad \Delta\psi_0 + \omega_0 = 0$$



QUASI-GEOSTROPHIC EQUATIONS

high-fidelity physical model constructed considering:

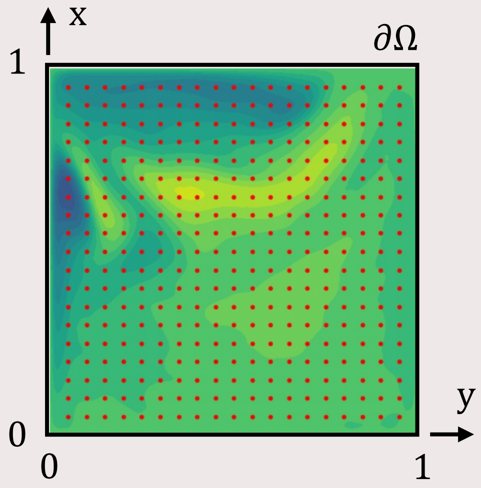
- mid-point discretization in time ($\Delta t = 0.1$)
- P1 finite elements discretization in space (3969 dofs)

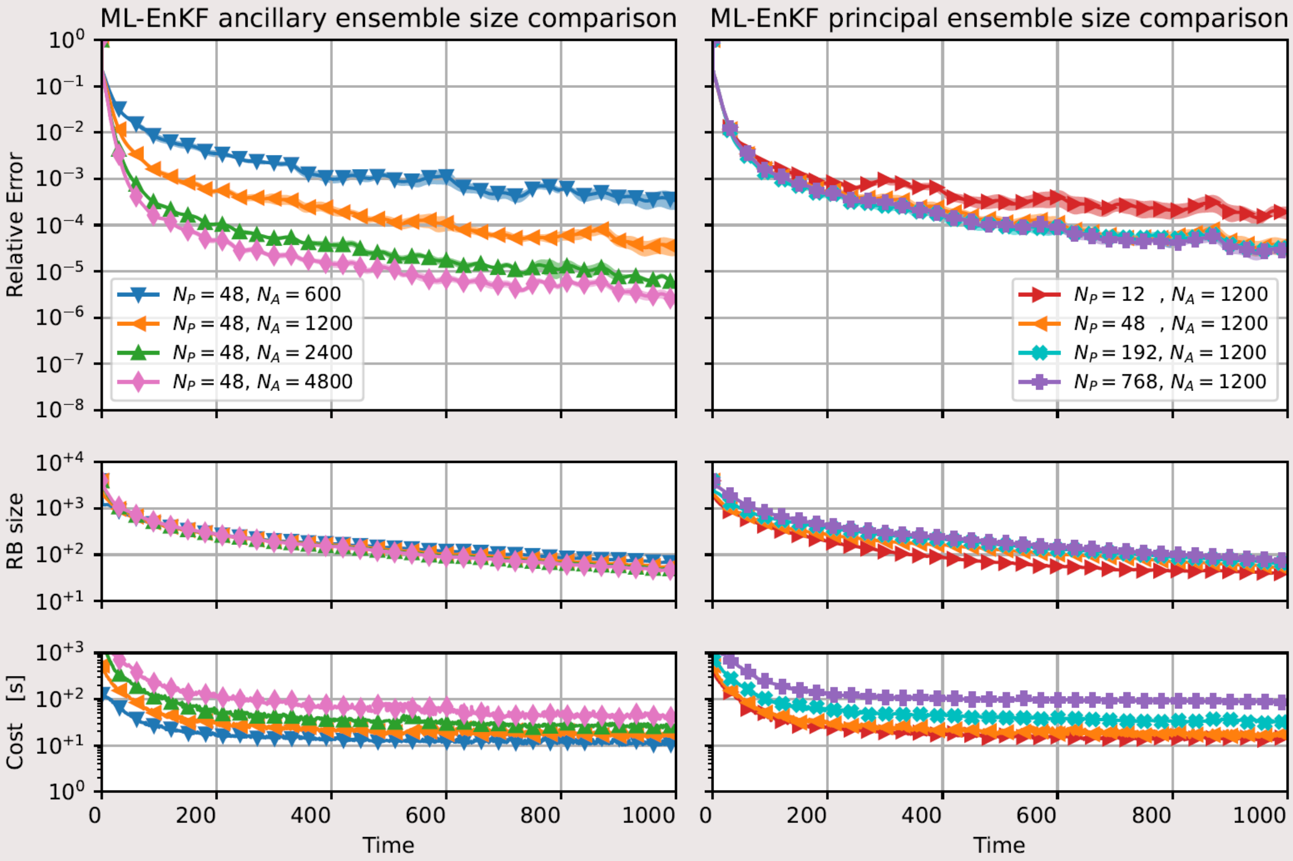
measurement model constructed considering:

- evenly spaced sensor positions (19x19)
- data collection every 10 time-steps

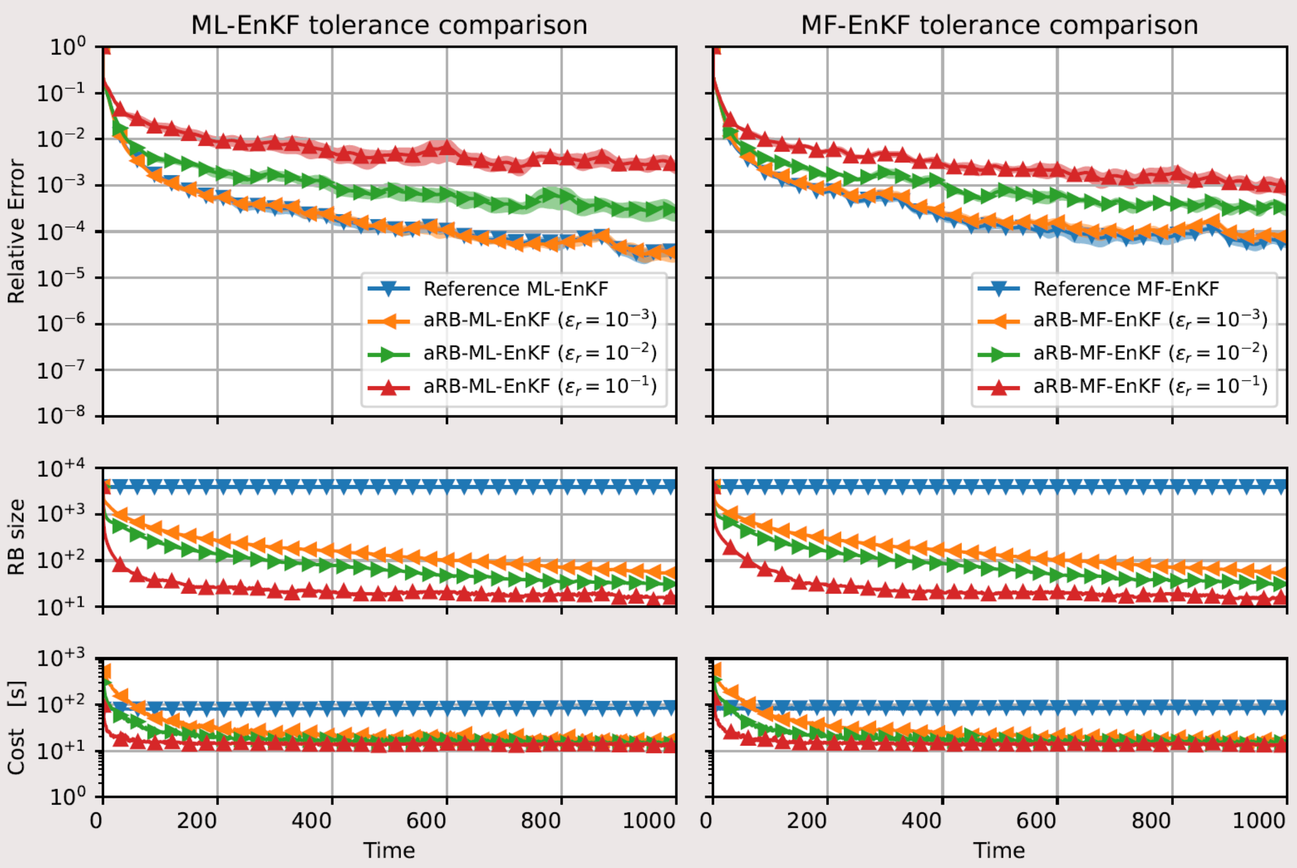
probabilistic model assumes:

- homoscedastic noise $\boldsymbol{\epsilon}_{n+1} \sim N(0, \sigma^2 \mathbf{I})$ $(\sigma = 10^{-4})$
- normal initial sample distribution $\omega_{0|-1} \sim N(0, \Delta^{-1})$





ML-EnKF ancillary ensemble size comparison
ML-EnKF principal ensemble size comparison
Relative Error
RB size
Cost [s]
Time
Time
$N_P = 48$, $N_A = 600$
$N_P = 48$, $N_A = 1200$
$N_P = 48$, $N_A = 2400$
$N_P = 48$, $N_A = 4800$
$N_P = 12$, $N_A = 1200$
$N_P = 48$, $N_A = 1200$
$N_P = 192$, $N_A = 1200$
$N_P = 768$, $N_A = 1200$



ML-EnKF tolerance comparison
MF-EnKF tolerance comparison
Relative Error
RB size
Cost [s]
Time
Reference ML-EnKF
aRB-ML-EnKF ($\varepsilon_r = 10^{-3}$)
aRB-ML-EnKF ($\varepsilon_r = 10^{-2}$)
aRB-ML-EnKF ($\varepsilon_r = 10^{-1}$)
Reference MF-EnKF
aRB-MF-EnKF ($\varepsilon_r = 10^{-3}$)
aRB-MF-EnKF ($\varepsilon_r = 10^{-2}$)
aRB-MF-EnKF ($\varepsilon_r = 10^{-1}$)

REFERENCES

[DM22] G. Donoghue and M. Yano. "**A multi-fidelity ensemble Kalman filter with hyperreduced reduced-order models**“. (2022)

[PMS21] A. Popov, C. Mou, A. Sandu, and T. Iliescu. "**A multifidelity ensemble Kalman filter with reduced order control variates.**" (2021)

[MWW20] C. Mou, Z. Wang, D. Wells, X. Xie and T. Iliescu. "**Reduced order models for the quasi-geostrophic equations: A brief survey.**" (2020)

[Che21] A. Chernov, H. Hoel, K.J.H. Law, F. Nobile, and R. Tempone. **”Multilevel ensemble Kalman filter for spatio-temporal processes."** (2021)

[Eve03] G. Evensen. "**The ensemble Kalman filter: Theoretical formulation and practical implementation**". (2003)

[Kal60] R. E. Kalman. "**A new approach to linear filtering and prediction problems**". (1960)

THANKS FOR YOUR ATTENTION!

f.a.b.silva@tue.nl