A multi-fidelity ensemble Kalman filter with adaptive reduced-order models

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ACKNOWLEDGMENTS

ERC-818473 : work supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program

DATA ASSIMILATION

DYNAMICAL MODEL

$$\omega_{\mathrm{n+1|n}} = \mathcal{M}\omega_{\mathrm{n|n}}$$

DATA ASSIMILATION

EXPERIMENTAL MEASUREMENTS

$$\boldsymbol{y}_{\mathrm{n+1}} = \mathbf{H}\omega_{\mathrm{T}} + \boldsymbol{\epsilon}_{\mathrm{n+1}}$$

DYNAMICAL MODEL

$$\omega_{\mathrm{n+1|n}} = \mathcal{M}\,\omega_{\mathrm{n|n}}$$

DATA ASSIMILATION

EXPERIMENTAL MEASUREMENTS

$$y_{n+1} = \mathbf{H}\omega_{T} + \epsilon_{n+1}$$

DA

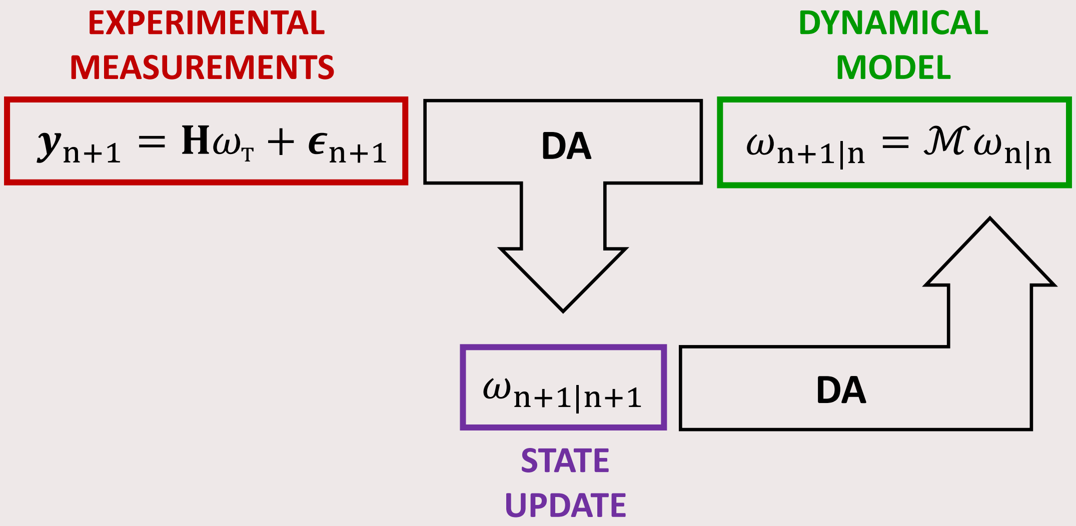
DYNAMICAL MODEL

$$\omega_{n+1|n} = \mathcal{M}\omega_{n|n}$$

$$\omega_{n+1|n+1}$$

STATE UPDATE

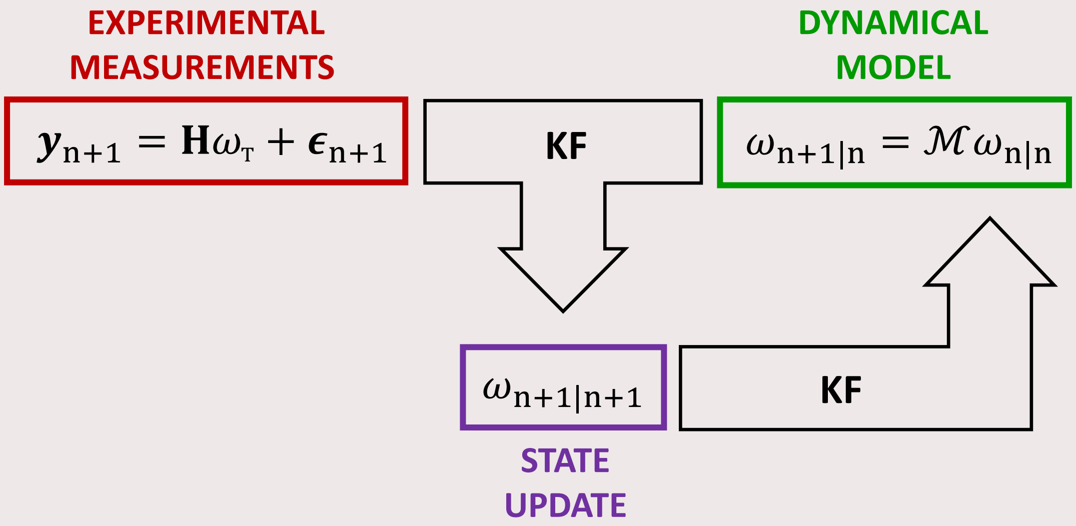
DATA ASSIMILATION



DATA ASSIMILATION

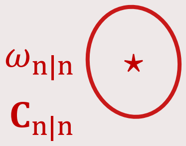
[Kal60]

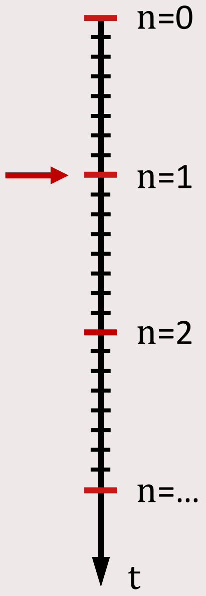
R. E. Kalman. "A new approach to linear filtering and prediction problems". (1960)



KALMAN FILTERING : PICTORIALLY

PRIOR





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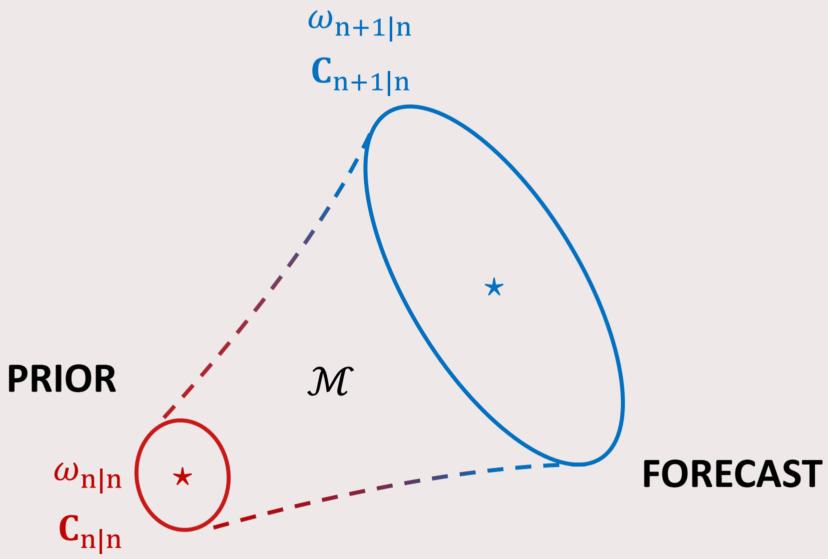

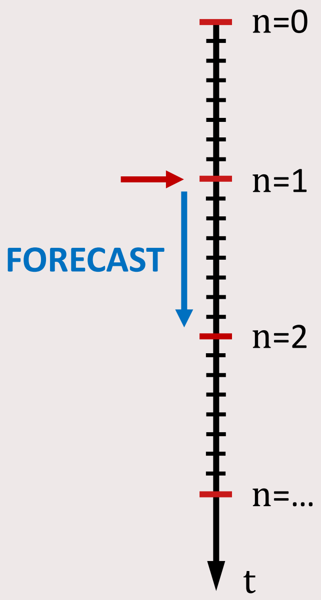

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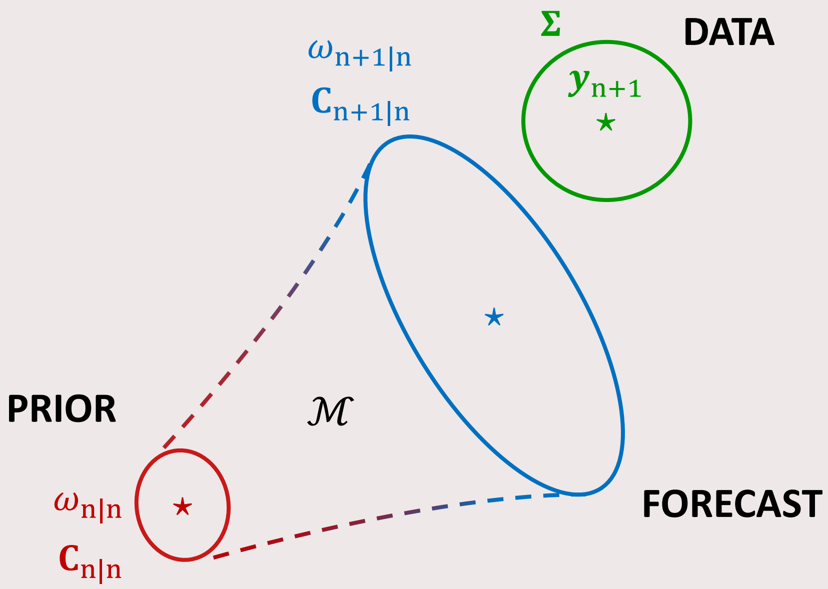

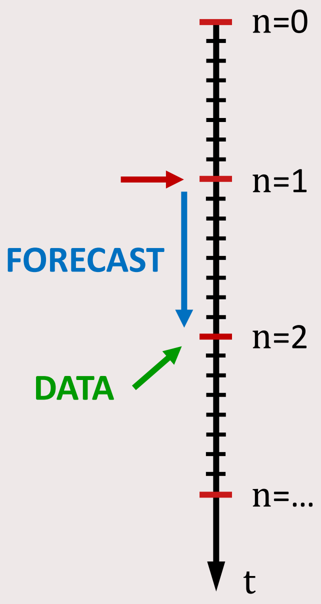

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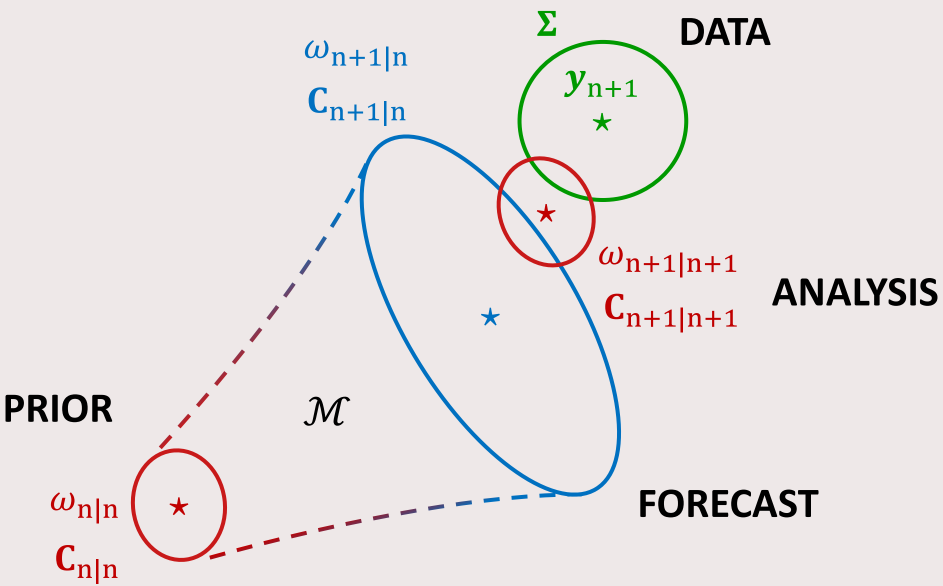
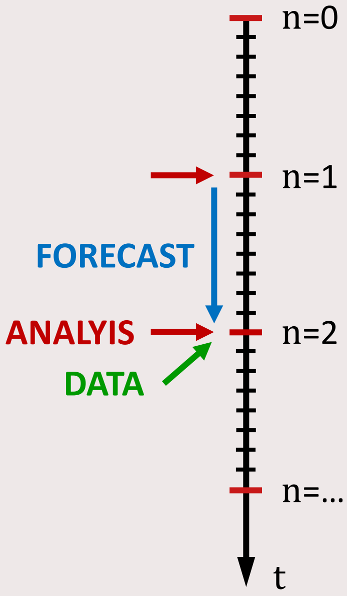
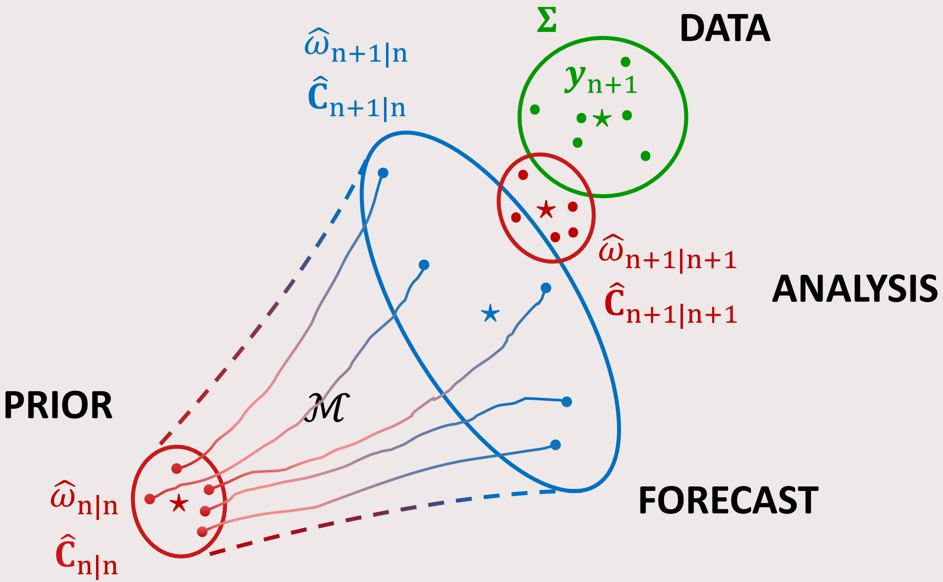


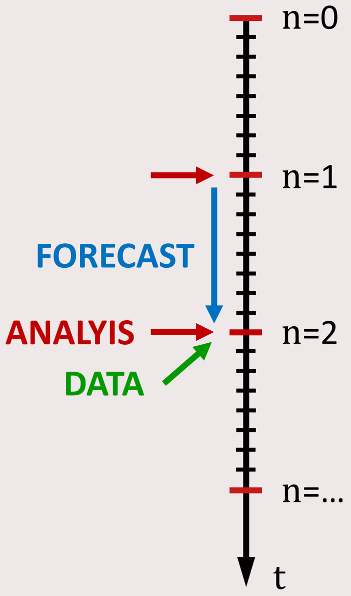


[Eve03]

G. Evensen. "The ensemble Kalman filter: Theoretical formulation and practical implementation". (2003)

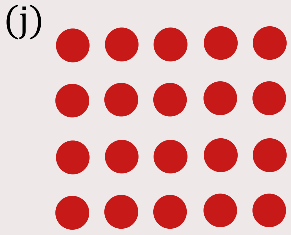
ENSEMBLE KALMAN FILTERING : PICTORIALLY





THE ENSEMBLE KALMAN FILTER

PREDICT : $\omega^{(j)}_{n+1|n} = \mathcal{M}\,\omega^{(j)}_{n|n}$

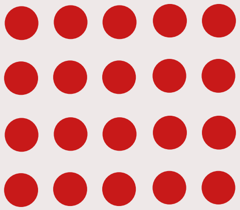


THE ENSEMBLE KALMAN FILTER

PREDICT : $\omega^{(j)}_{n+1|n} = \mathcal{M}\,\omega^{(j)}_{n|n}$

ESTIMATE : $\hat{\mathbf{C}}_{n+1|n} = \mathrm{cov}\left\{\omega^{(j)}_{n+1|n}\right\}$

(j)



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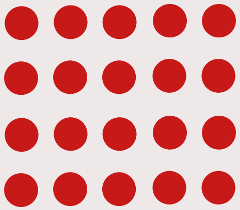
ANALYSE : $\omega^{(j)}_{n+1|n+1} = \omega^{(j)}_{n+1|n} + \widehat{\mathbf{K}}_{n+1}\left(\boldsymbol{y}_{n+1} - \mathbf{H}\omega^{(j)}_{n+1|n}\right)$

(j)



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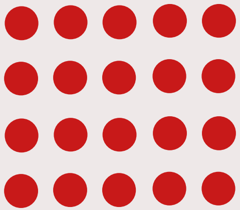


employing the empirical Kalman gain

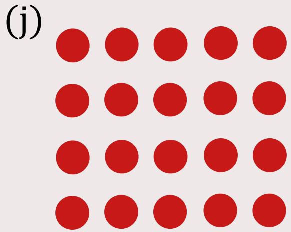
$$\widehat{\mathbf{K}}_{n+1} = \hat{\mathbf{C}}_{n+1|n}\mathbf{H}^*(\mathbf{H}\hat{\mathbf{C}}_{n+1|n}\mathbf{H}^* + \boldsymbol{\Sigma})^{-1}$$

THE ENSEMBLE KALMAN FILTER

PREDICT : $\omega^{(j)}_{n+1|n} = \mathcal{M}\,\omega^{(j)}_{n|n}$ ← **EXPENSIVE!**

ESTIMATE : $\hat{\mathbf{C}}_{n+1|n} = \mathrm{cov}\left\{\omega^{(j)}_{n+1|n}\right\}$

ANALYSE : $\omega^{(j)}_{n+1|n+1} = \omega^{(j)}_{n+1|n} + \widehat{\mathbf{K}}_{n+1}\left(\boldsymbol{y}_{n+1} - \mathbf{H}\omega^{(j)}_{n+1|n}\right)$



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[PMS21]

A. Popov, et all. "A multifidelity ensemble Kalman filter with reduced order control variates." (2021)

THE ENSEMBLE KALMAN FILTER

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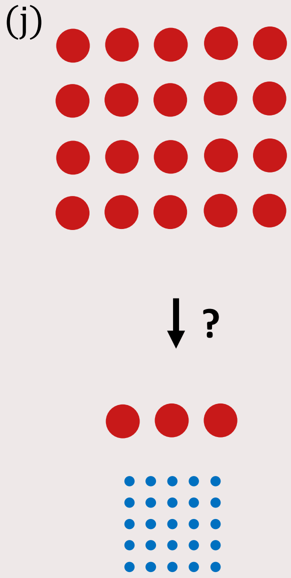
ANALYSE : $\omega^{(j)}_{n+1|n+1} = \omega^{(j)}_{n+1|n} + \widehat{\mathbf{K}}_{n+1}\left(y_{n+1} - \mathbf{H}\omega^{(j)}_{n+1|n}\right)$

employing the empirical Kalman gain

$$\widehat{\mathbf{K}}_{n+1} = \hat{\mathbf{C}}_{n+1|n}\mathbf{H}^*(\mathbf{H}\hat{\mathbf{C}}_{n+1|n}\mathbf{H}^* + \mathbf{\Sigma})^{-1}$$

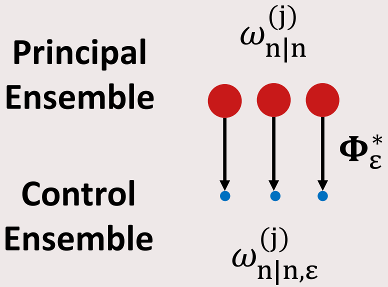


THE MULTI-FIDELITY ENSEMBLE KALMAN FILTER : SOME COMPLICATION

Principal Ensemble

$\omega_{n|n}^{(j)}$

THE MULTI-FIDELITY ENSEMBLE KALMAN FILTER : SOME COMPLICATION



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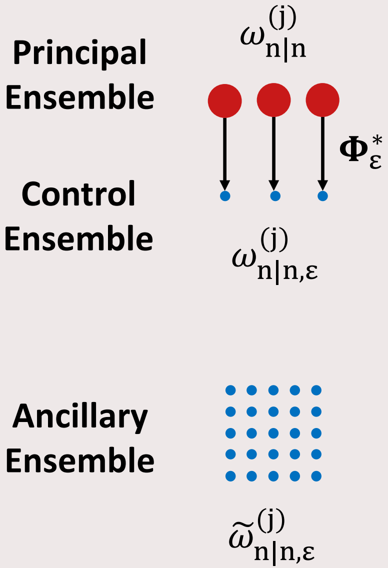

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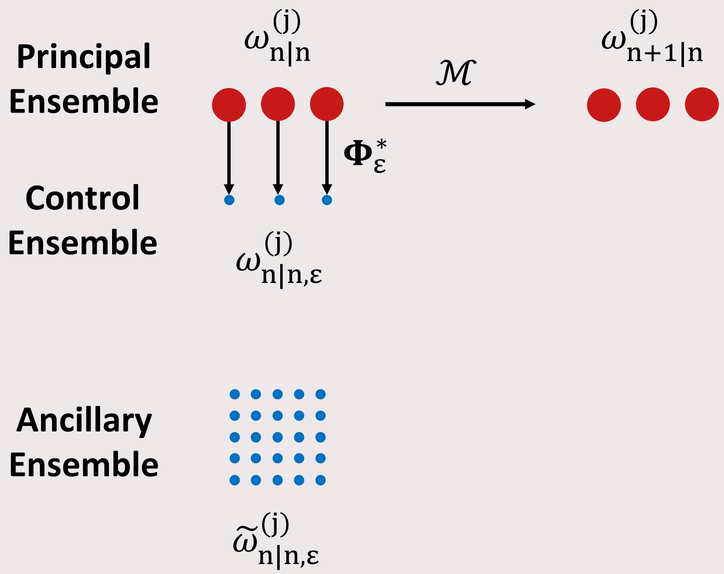

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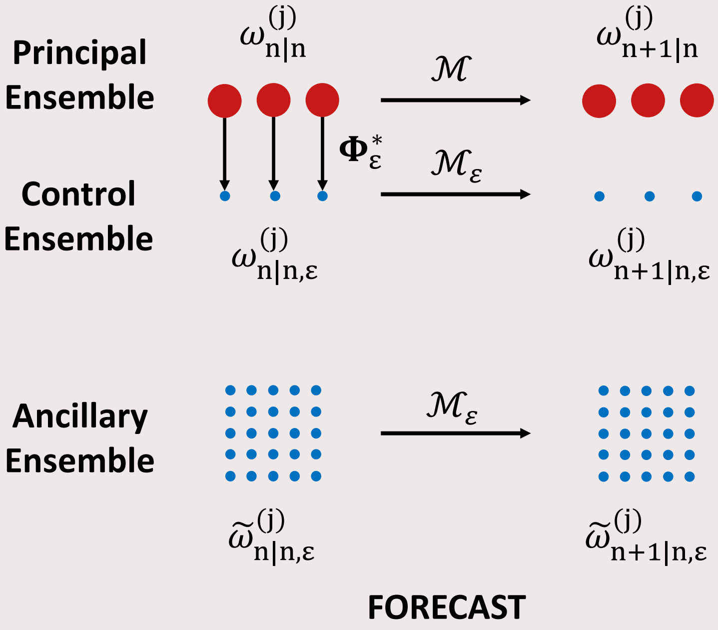

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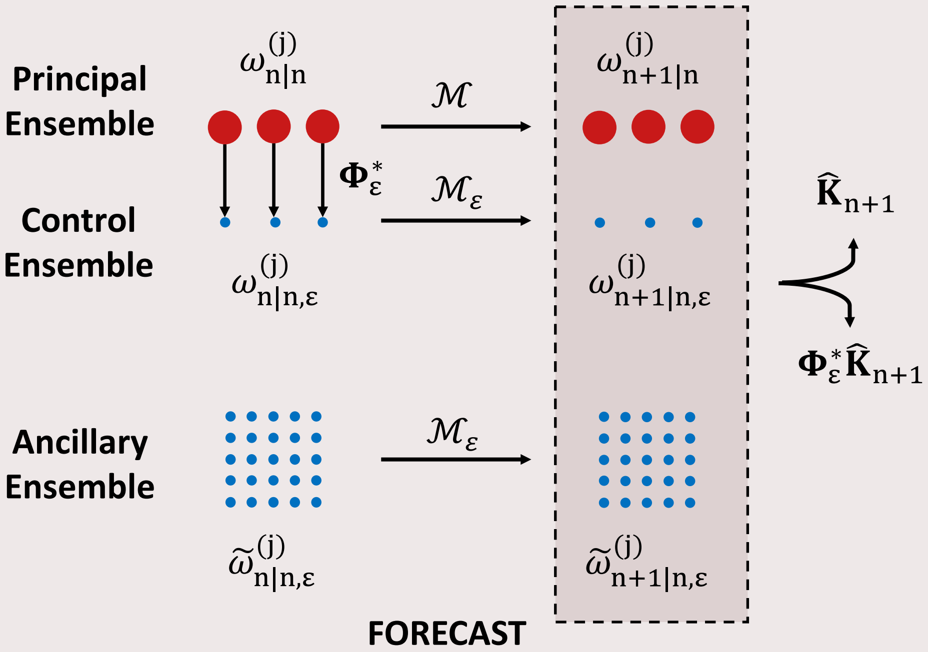

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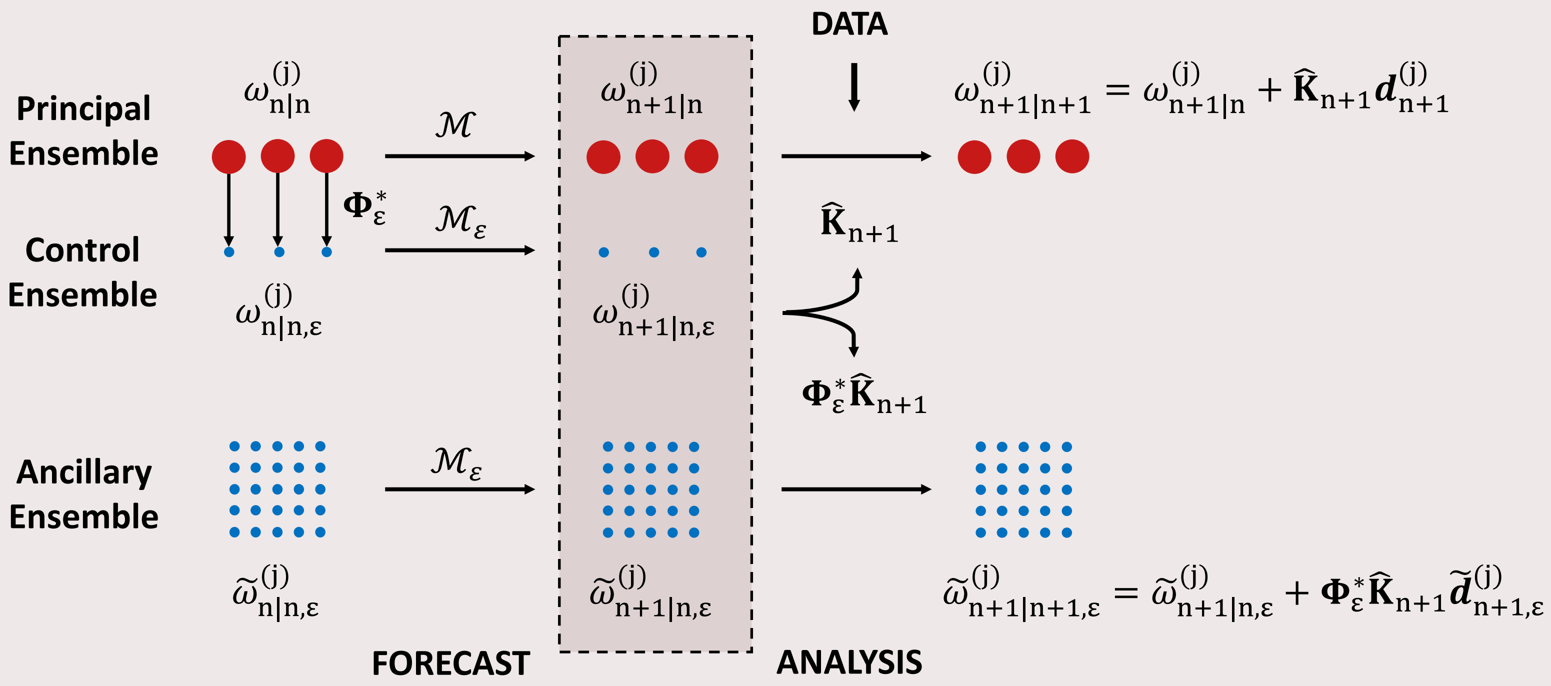

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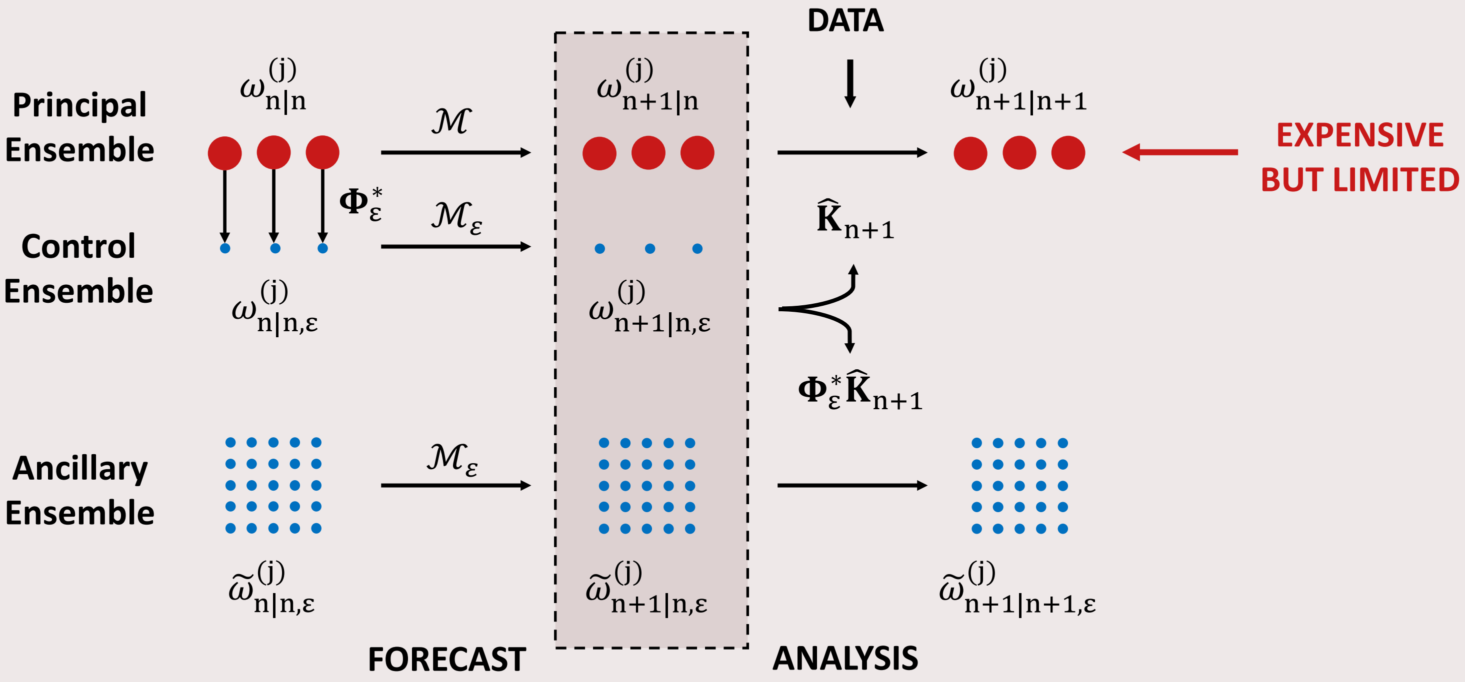

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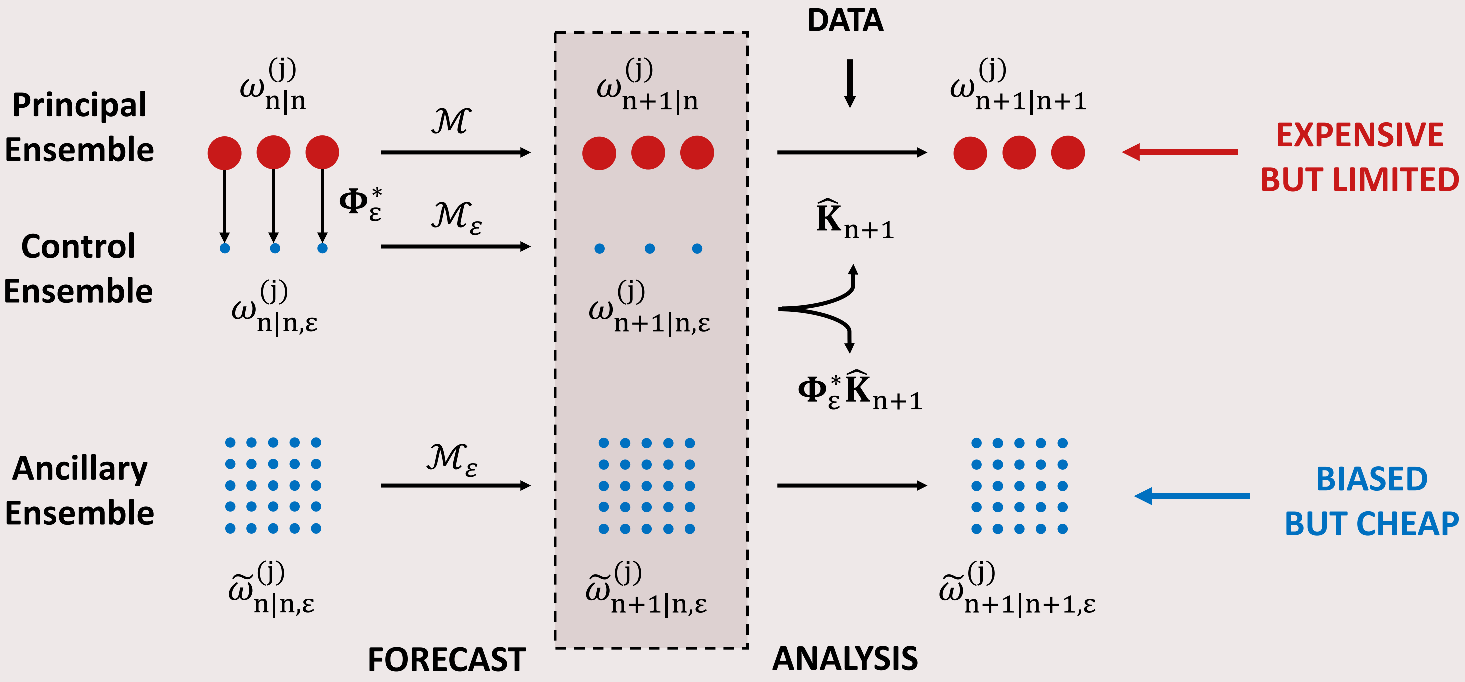

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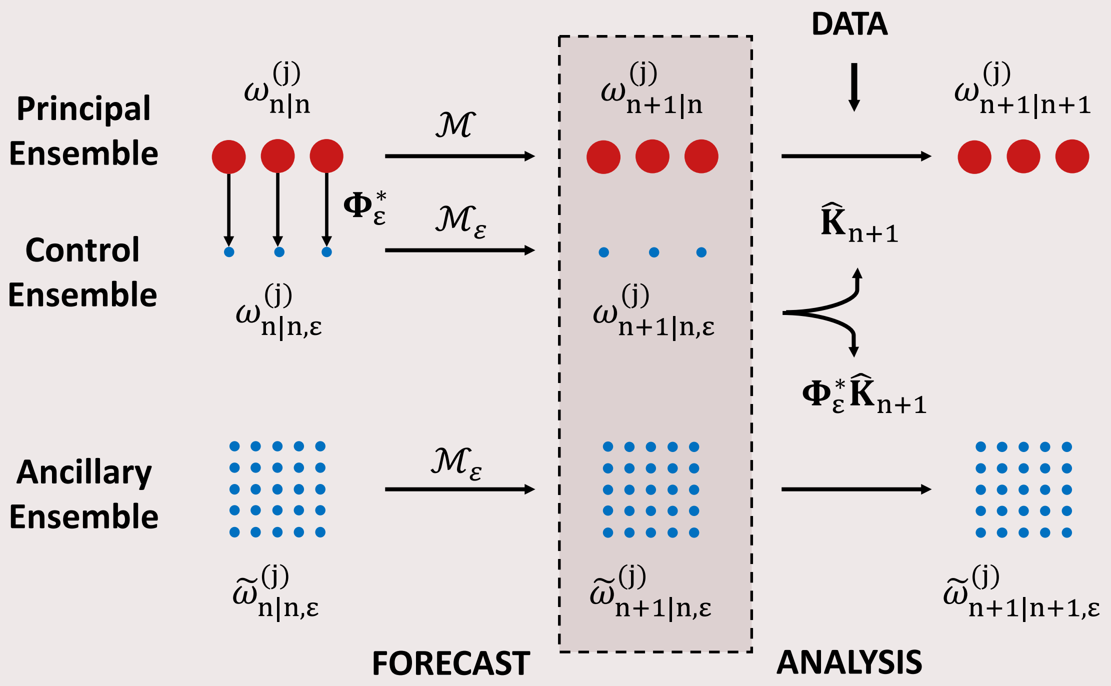


? $\mathcal{M}_\varepsilon$?

[DM22]

G. Donoghue and M. Yano. "A multi-fidelity ensemble Kalman filter with hyperreduced reduced-order models". (2022)

REDUCED ORDER MODEL : DONOGHUE AND YANO

Principal Ensemble

$$\omega_{n|n}^{(j)} = \omega_0^{(j)} \xrightarrow{\mathcal{M}_{\Delta t}} \omega_1^{(j)} \xrightarrow{\dots} \omega_{N-1}^{(j)} \xrightarrow{\mathcal{M}_{\Delta t}} \omega_N^{(j)} = \omega_{n+1|n}^{(j)}$$

$$\mathcal{M} = \mathcal{M}_{\Delta t} \circ \dots \circ \mathcal{M}_{\Delta t}$$

$$\mathcal{W}_\varepsilon = \mathrm{POD}\left(\cup_{r=1}^{N}\left\{\omega_r^{(j)}\right\}\right) \longrightarrow \begin{array}{l} \mathbf{\Phi}_\varepsilon : \mathcal{W} \to \mathcal{W}_\varepsilon \\ \mathbf{\Phi}_\varepsilon^* : \mathcal{W}_\varepsilon \to \mathcal{W} \end{array} \longrightarrow \begin{array}{l} \mathcal{M}_{\Delta t,\varepsilon} = \mathbf{\Phi}_\varepsilon^* \circ \mathcal{M}_{\Delta t} \circ \mathbf{\Phi}_\varepsilon \\ \mathcal{M}_\varepsilon = \mathcal{M}_{\Delta t,\varepsilon} \circ \dots \circ \mathcal{M}_{\Delta t,\varepsilon} \end{array}$$

REDUCED ORDER MODEL : ADAPTIVE GREEDY COSTRUCTION

Principal Ensemble

$\omega_{n|n}^{(j)}$

REDUCED ORDER MODEL : ADAPTIVE GREEDY COSTRUCTION

Principal Ensemble

$\omega_{n|n}^{(j)}$

$$\mathcal{W}_{\varepsilon}^{0} = \mathrm{POD}\left(\left\{\omega_{n|n}^{(j)}\right\}\right)$$

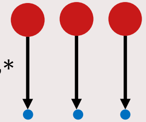
REDUCED ORDER MODEL : ADAPTIVE GREEDY COSTRUCTION

Principal Ensemble $\omega_{n|n}^{(j)}$ ● ● ●

$$\mathcal{W}_\varepsilon^0 = \mathrm{POD}\left(\left\{\omega_{n|n}^{(j)}\right\}\right) \longrightarrow \begin{array}{l} \mathbf{\Phi}_\varepsilon^0 : \mathcal{W} \to \mathcal{W}_\varepsilon^0 \\ \mathbf{\Phi}_\varepsilon^{*,0} : \mathcal{W}_\varepsilon^0 \to \mathcal{W} \end{array}$$

REDUCED ORDER MODEL : ADAPTIVE GREEDY COSTRUCTION

Principal Ensemble $\omega_{n|n}^{(j)}$

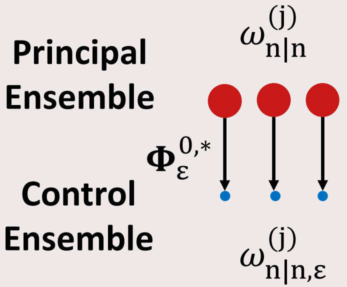


$\boldsymbol{\Phi}_{\varepsilon}^{0,*}$

Control Ensemble $\omega_{n|n,\varepsilon}^{(j)}$

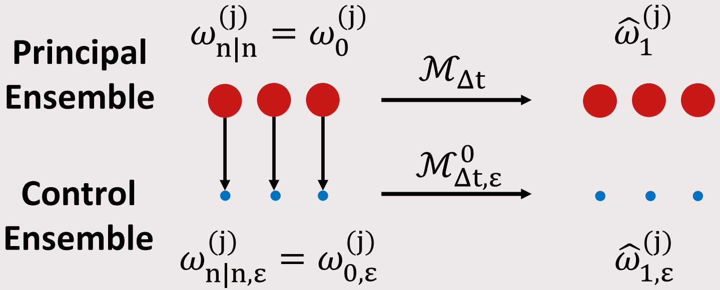
$$\mathcal{W}_{\varepsilon}^{0} = \mathrm{POD}\left(\left\{\omega_{n|n}^{(j)}\right\}\right) \longrightarrow \begin{array}{l} \boldsymbol{\Phi}_{\varepsilon}^{0} \;:\; \mathcal{W} \to \mathcal{W}_{\varepsilon}^{0} \\ \boldsymbol{\Phi}_{\varepsilon}^{*,0} \;:\; \mathcal{W}_{\varepsilon}^{0} \to \mathcal{W} \end{array}$$

REDUCED ORDER MODEL : ADAPTIVE GREEDY COSTRUCTION



$$\mathcal{W}_{\varepsilon}^{0} = \mathrm{POD}\left(\left\{\omega_{n|n}^{(j)}\right\}\right) \longrightarrow \begin{array}{l} \mathbf{\Phi}_{\varepsilon}^{0} : \mathcal{W} \to \mathcal{W}_{\varepsilon}^{0} \\ \mathbf{\Phi}_{\varepsilon}^{*,0} : \mathcal{W}_{\varepsilon}^{0} \to \mathcal{W} \end{array} \longrightarrow \mathcal{M}_{\Delta t,\varepsilon}^{0} = \mathbf{\Phi}_{\varepsilon}^{*} \circ \mathcal{M}_{\Delta t} \circ \mathbf{\Phi}_{\varepsilon}$$

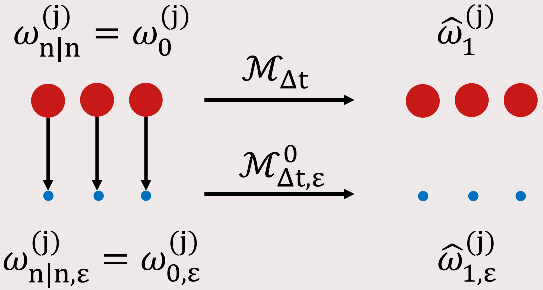
REDUCED ORDER MODEL : ADAPTIVE GREEDY COSTRUCTION



REDUCED ORDER MODEL : ADAPTIVE GREEDY COSTRUCTION

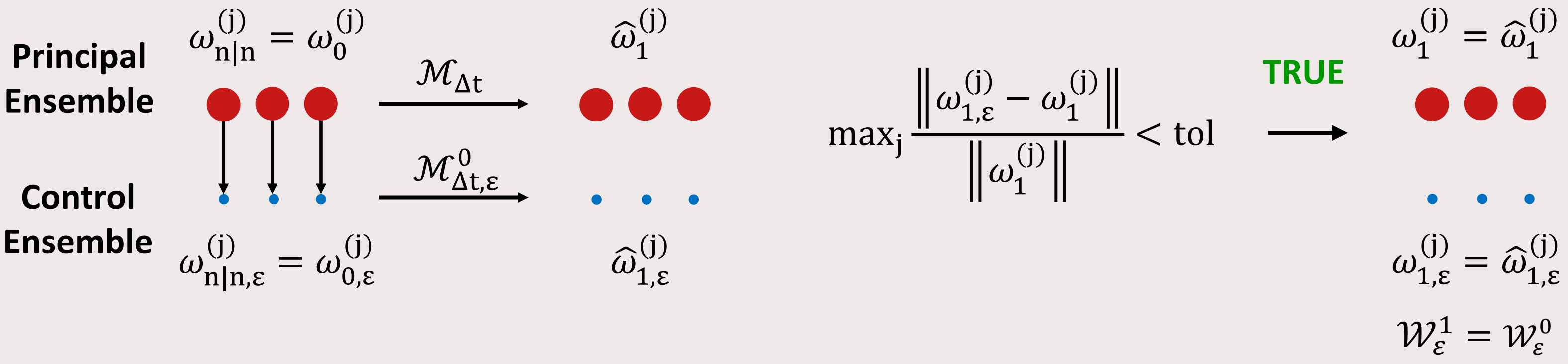
Principal Ensemble

Control Ensemble



$$\max_j \frac{\left\| \omega_{1,\varepsilon}^{(j)} - \omega_1^{(j)} \right\|}{\left\| \omega_1^{(j)} \right\|} < \mathrm{tol}$$

REDUCED ORDER MODEL : ADAPTIVE GREEDY COSTRUCTION



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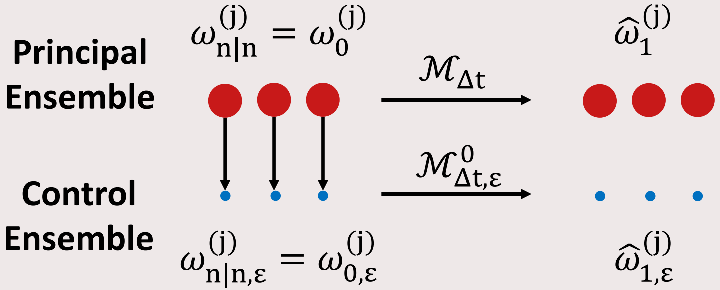


$$\max_j \frac{\left\| \omega_{1,\varepsilon}^{(j)} - \omega_1^{(j)} \right\|}{\left\| \omega_1^{(j)} \right\|} < \text{tol}$$

TRUE $\longrightarrow$

$$\omega_1^{(j)} = \widehat{\omega}_1^{(j)}$$

$$\omega_{1,\varepsilon}^{(j)} = \widehat{\omega}_{1,\varepsilon}^{(j)}$$

$$\mathcal{W}_\varepsilon^1 = \mathcal{W}_\varepsilon^0$$

FALSE $\downarrow$

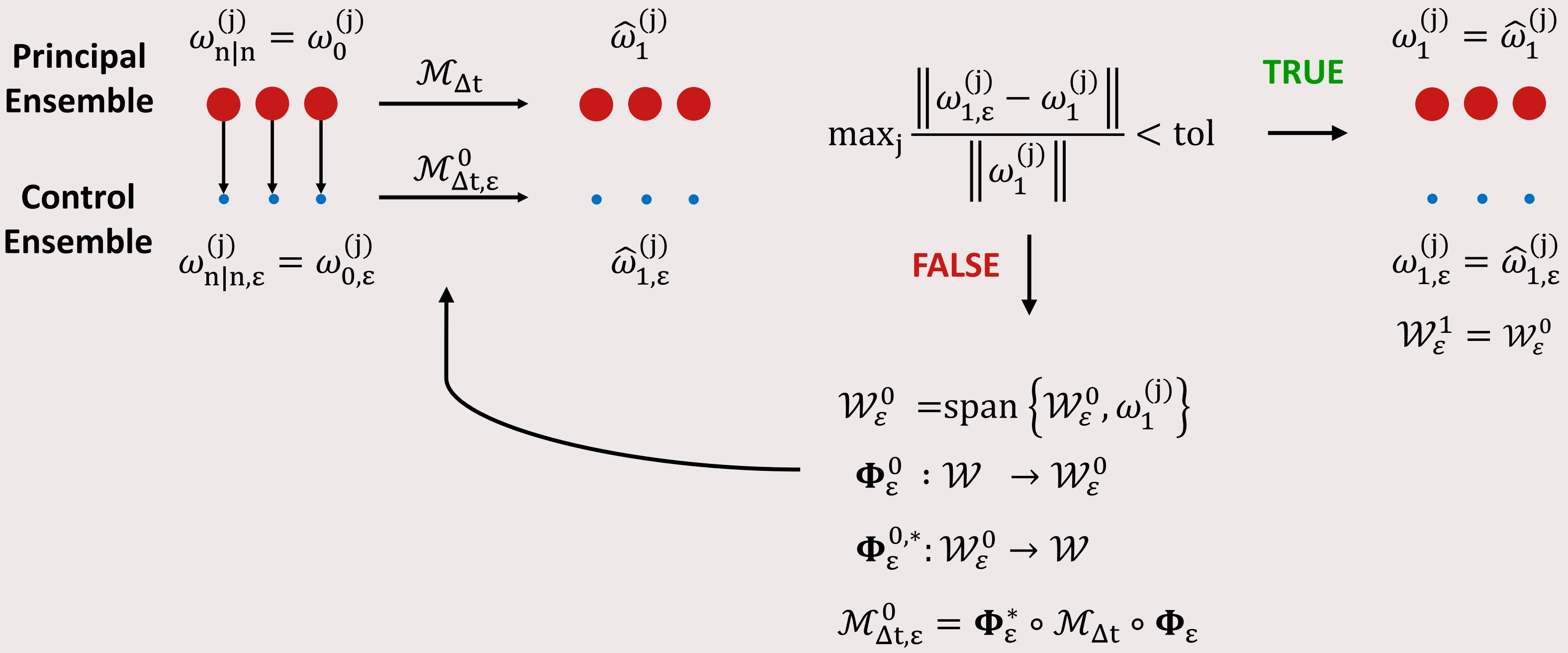
$$\mathcal{W}_\varepsilon^0 = \text{span}\left\{ \mathcal{W}_\varepsilon^0, \omega_1^{(j)} \right\}$$

$$\mathbf{\Phi}_\varepsilon^0 : \mathcal{W} \rightarrow \mathcal{W}_\varepsilon^0$$

$$\mathbf{\Phi}_\varepsilon^{0,*} : \mathcal{W}_\varepsilon^0 \rightarrow \mathcal{W}$$

$$\mathcal{M}_{\Delta t,\varepsilon}^0 = \mathbf{\Phi}_\varepsilon^* \circ \mathcal{M}_{\Delta t} \circ \mathbf{\Phi}_\varepsilon$$

REDUCED ORDER MODEL : ADAPTIVE GREEDY COSTRUCTION



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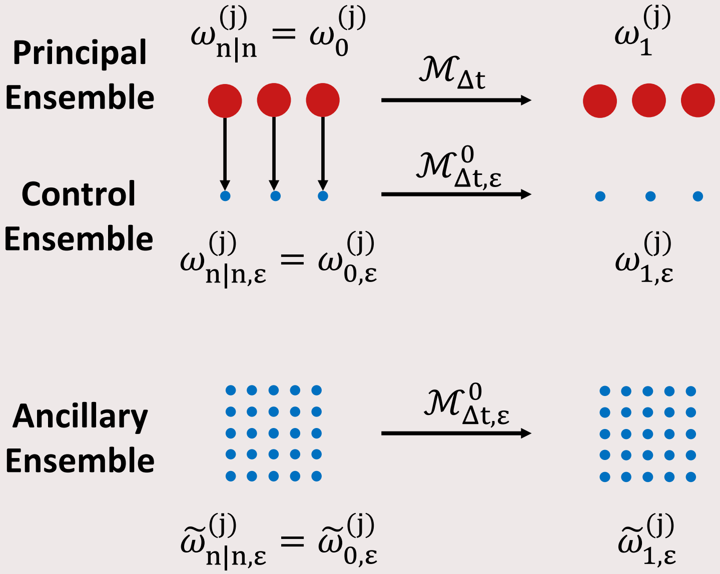

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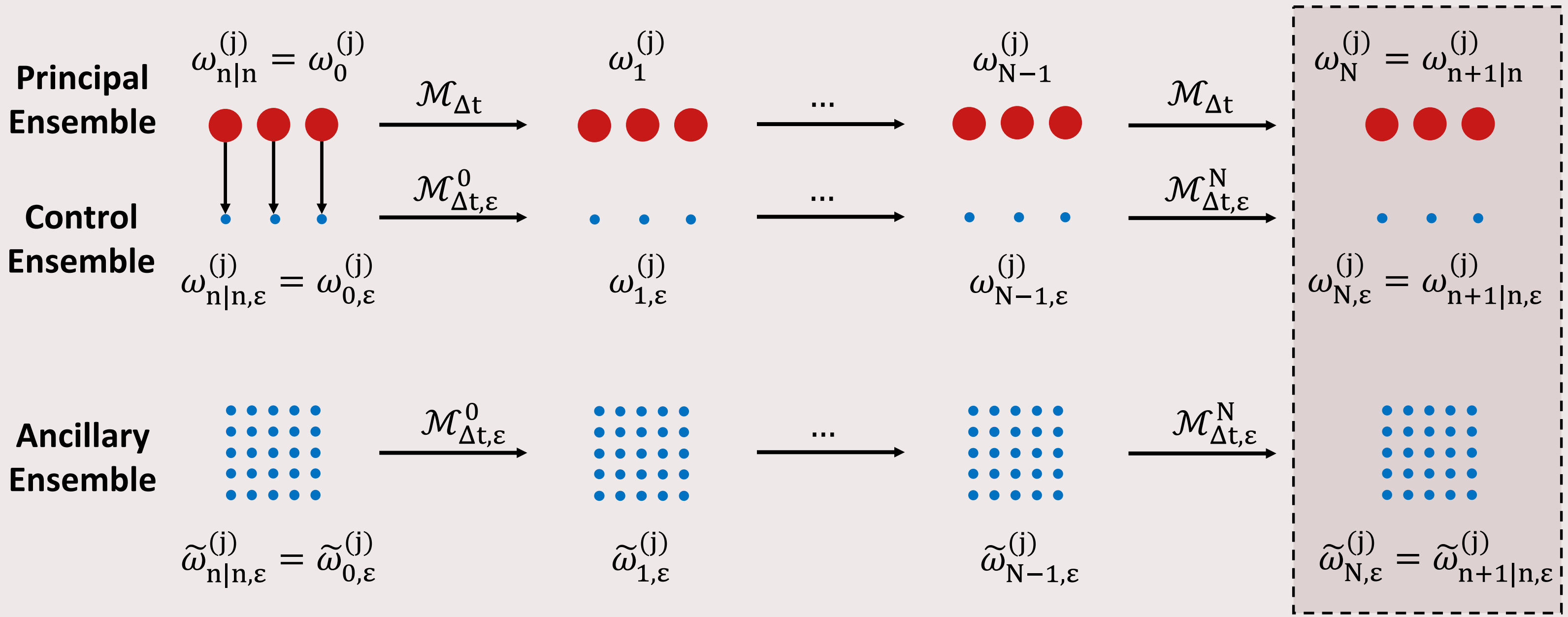


[MWW20]
C. Mou, Z. Wang, D. Wells, X. Xie and T. Iliescu. "Reduced order models for the quasi-geostrophic equations: A brief survey." (2020)

QUASI-GEOSTROPHIC EQUATIONS

find $\omega = \omega(x,y,t)$, $\psi = \psi(x,y,t)$ such that

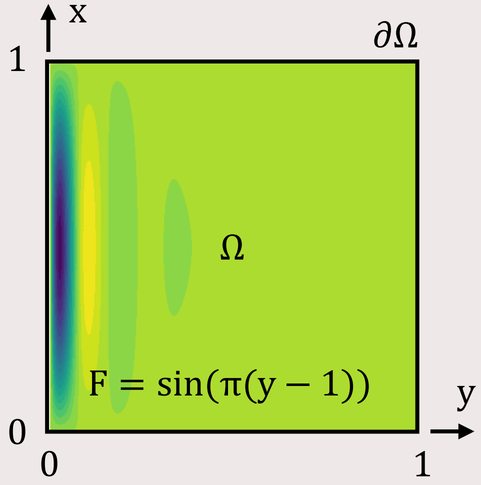
$$\partial_t \omega = \mathrm{Ro}\, J(\omega,\psi) + \partial_x \psi + \frac{\mathrm{Ro}}{\mathrm{Re}} \Delta\omega + F, 0 \Delta\psi + \omega = 0 \longleftarrow J(\omega,\psi) = \partial_x \psi \partial_y \omega - \partial_x \psi \partial_y \omega$$

given the boundary and initial conditions

$$\omega(x,y,t) = 0, \quad (x,y) \in \partial\Omega$$
$$\psi(x,y,t) = 0, \quad (x,y) \in \partial\Omega$$
$$\omega(x,y,0) = \omega_0, \quad (x,y) \in \Omega$$

and

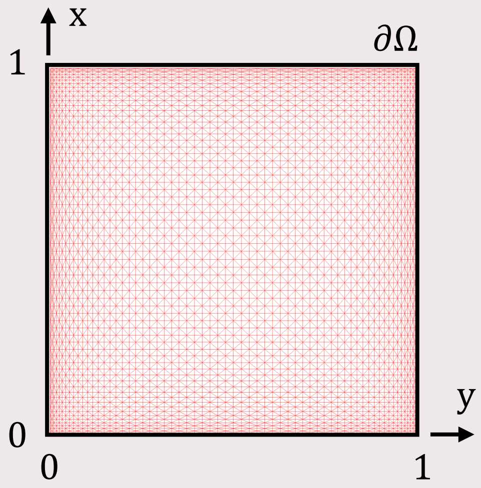
$$\partial_x \psi_0 + \frac{\mathrm{Ro}}{\mathrm{Re}} \Delta\omega_0 + F = 0, \; \Delta\psi_0 + \omega_0 = 0$$



QUASI-GEOSTROPHIC EQUATIONS

high-fidelity physical model constructed considering:

- fully implicit Crank-Nicholson discretization in time $(\mathrm{dt} = 0.1)$
- P1 finite elements discretization in space $(4225\ \mathrm{dofs})$



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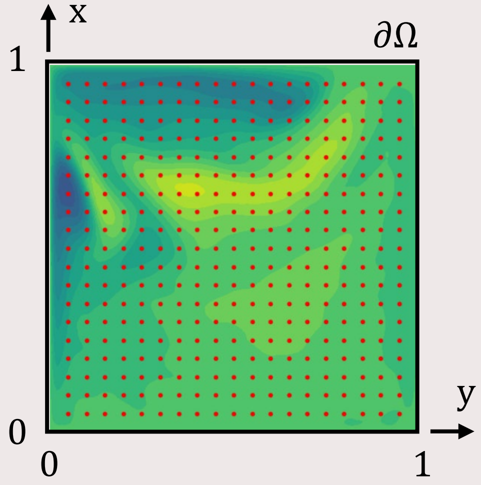
measurement model constructed considering:

- evenly spaced sensor positions $(19\mathrm{x}19)$
- data collection every 10 time-steps



QUASI-GEOSTROPHIC EQUATIONS

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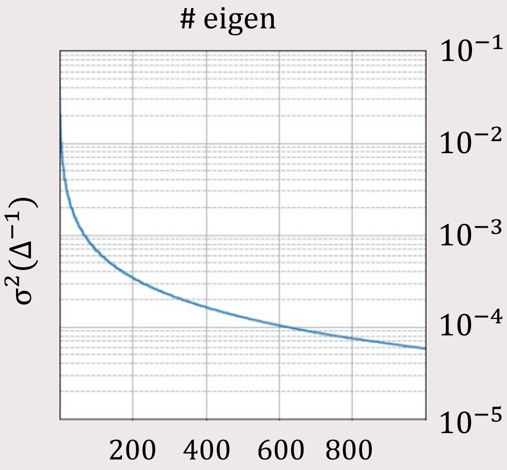
- P1 finite elements discretization in space (4225 dofs)

measurement model constructed considering:

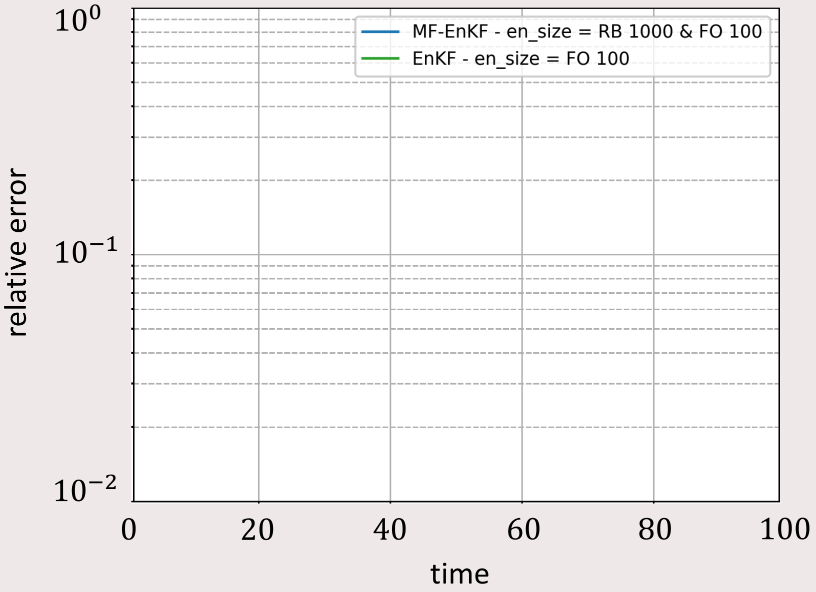
- evenly spaced sensor positions ($19x19$)
- data collection every 10 time-steps

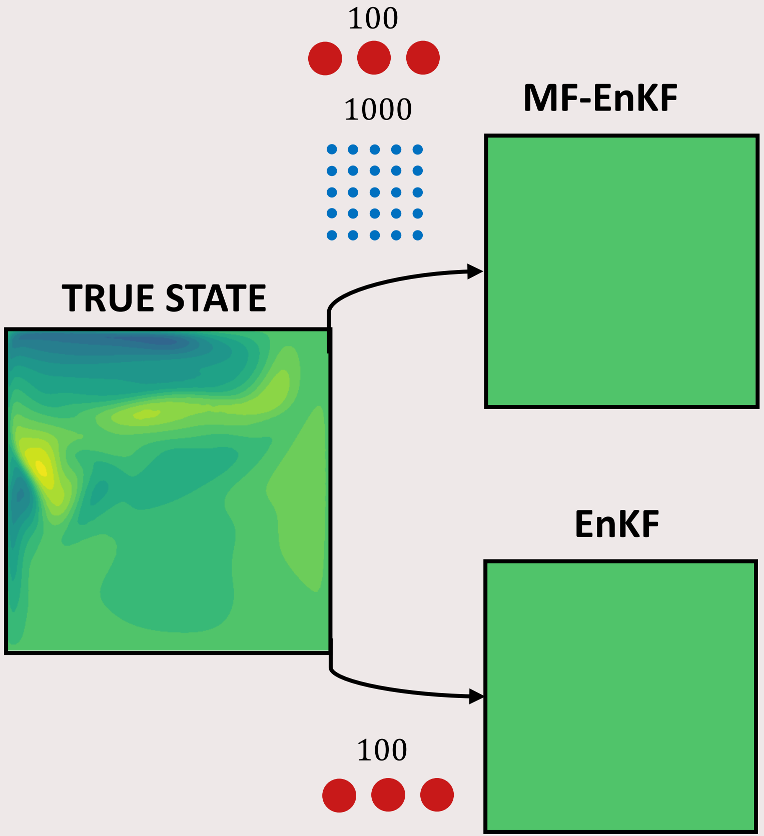
probabilistic model assumes:

- homoscedastic noise $\boldsymbol{\epsilon}_{n+1} \sim N(0, \sigma^2 \mathbf{I})$ ($\sigma = 10^{-4}$)
- normal initial sample distribution $\omega_{0|0} \sim N(0, \Delta^{-1})$



QUASI-GEOSTROPHIC EQUATIONS





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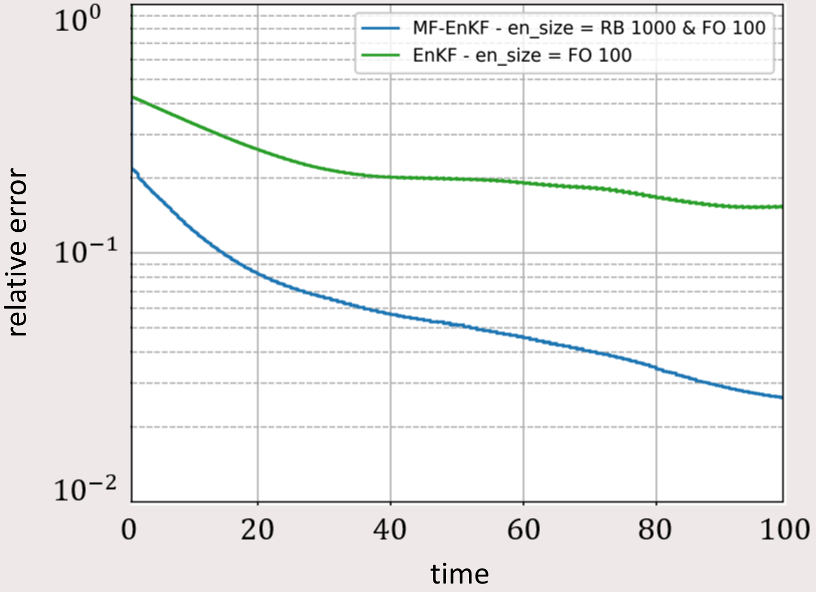

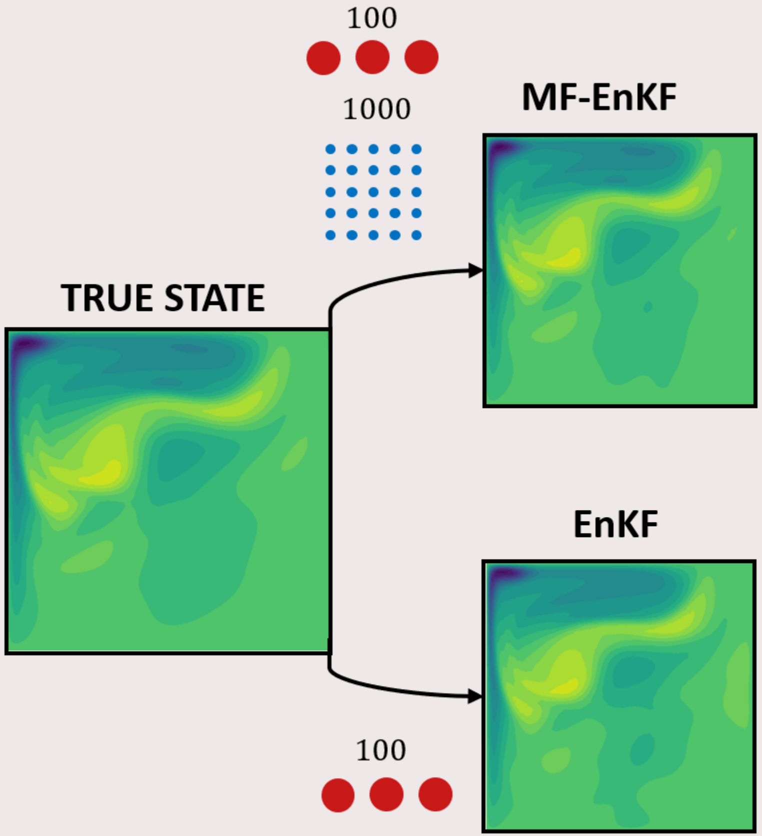

CONCLUSIONS

SUMMARY :

- we developed an adaptive RB method for the MF-EnKF
- we performed preliminary tests using a quasi-geostrophic model

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OUTLOOK :

- we plan to implement evolution equations for the RB space
- we plan to carry out further experiments with the presented model

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THANKS FOR YOUR ATTENTION!