



The Reduced Basis Ensemble Kalman Method : an Iterative Regularization Method

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ACKNOWLEDGMENTS

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OUTLINE

1. INTRODUCTION
2. VARIATIONAL METHODS
3. MODEL APPROXIMATION
4. THE REDUCED BASIS ENSEMBLE KALMAN METHOD
5. NUMERICAL EXPERIMENTS
6. CONCLUSIONS

ASYNCHRONOUS DATA ASSIMILATION : AN INVERSE PROBLEM

$$\mathbf{y} = \mathbf{L}u_{\text{TRUE}} + \epsilon$$

**ASYNCHRONOUS
MEASUREMENTS**

ASYNCHRONOUS DATA ASSIMILATION : AN INVERSE PROBLEM

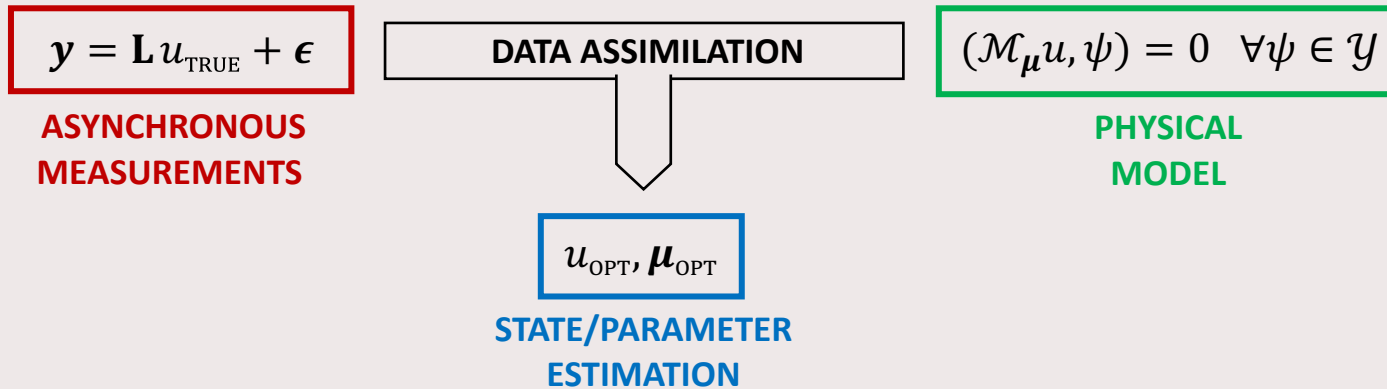
$$\mathbf{y} = \mathbf{L}u_{\text{TRUE}} + \boldsymbol{\epsilon}$$

**ASYNCHRONOUS
MEASUREMENTS**

$$(\mathcal{M}_{\mu}u, \psi) = 0 \quad \forall \psi \in \mathcal{Y}$$

**PHYSICAL
MODEL**

ASYNCHRONOUS DATA ASSIMILATION : AN INVERSE PROBLEM



[SEB10]

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VARIATIONAL DATA ASSIMILATION : CONSTRAINED MINIMIZATION

$$\min_{\mu \in \mathcal{P}} \mathcal{J}(\mu | \mathbf{y}) := \frac{1}{2} \|\mathbf{y} - \mathbf{L}u\|_{\Sigma^{-1}}^2$$

DATA MISFIT

such that

$$(\mathcal{M}_\mu u, \psi) = 0 \quad \forall \psi \in \mathcal{Y}$$

WEAK MODEL

where:

$$\mathbf{y} = \mathbf{L}u_{\text{TRUE}} + \epsilon \quad \text{with noise} \quad \epsilon \sim \mathcal{N}(0, \Sigma)$$

VARIATIONAL DATA ASSIMILATION : REGULARIZED

$$\min_{\mu \in \mathcal{P}} \mathcal{J}(\mu | \mathbf{y}) := \frac{1}{2} \boxed{\|\mathbf{y} - \mathbf{L}u\|_{\Sigma^{-1}}^2} + \boxed{\mathcal{J}(\mu)} \quad \text{such that} \quad \boxed{(\mathcal{M}_\mu u, \psi) = 0 \quad \forall \psi \in \mathcal{Y}}$$

DATA MISFIT **STABILIZATION** **WEAK MODEL**

where:

$$\mathbf{y} = \mathbf{L}u_{\text{TRUE}} + \epsilon \quad \text{with noise} \quad \epsilon \sim \mathcal{N}(0, \Sigma)$$

[TC91]

VARIATIONAL DATA ASSIMILATION : UNREGULARIZED

$$\min_{\boldsymbol{\mu} \in \mathcal{P}} \mathcal{J}(\boldsymbol{\mu} | \mathbf{y}) := \frac{1}{2} \|\mathbf{y} - \mathbf{L}u\|_{\Sigma^{-1}}^2$$

DATA MISFIT

such that

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WEAK MODEL

where:

$$\mathbf{y} = \mathbf{L}u_{\text{TRUE}} + \boldsymbol{\epsilon} \quad \text{with noise } \boldsymbol{\epsilon} \sim \mathcal{N}(0, \Sigma)$$

the solution of the un-regularized problem can be obtained employing an iterative regularization methods

$$\boldsymbol{\mu}_{k+1} = \boldsymbol{\mu}_k + \mathcal{G}_k(\boldsymbol{\mu}_k, \mathbf{y}) \quad \longleftarrow \text{Landweber iterations}$$

[KNS08]

[Lan51]

VARIATIONAL DATA ASSIMILATION : UNREGULARIZED

$$\min_{\mu \in \mathcal{P}} \mathcal{J}(\mu | \mathbf{y}) := \frac{1}{2} \|\mathbf{y} - \mathbf{L}u\|_{\Sigma^{-1}}^2$$

DATA MISFIT

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the solution of the un-regularized problem can be obtained employing an iterative regularization methods; those can be implemented via



Local approaches (Newton's type methods)

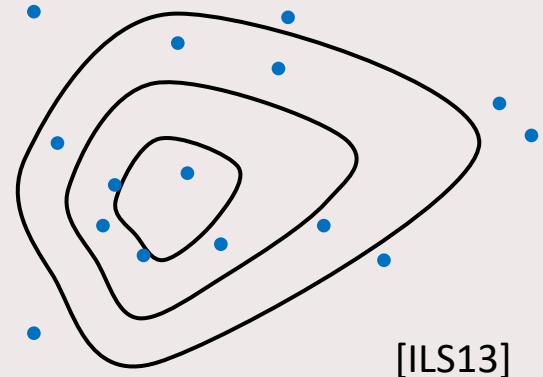


Global approaches (Particles based methods)

THE ENSEMBLE KALMAN METHOD

We sample a particle ensemble of size J from a prior distribution π_0 and update their positions as follows:

$$\bullet \mu_0^{(j)} \sim \pi_0 := e^{-\mathcal{J}(\mu)}$$



THE ENSEMBLE KALMAN METHOD

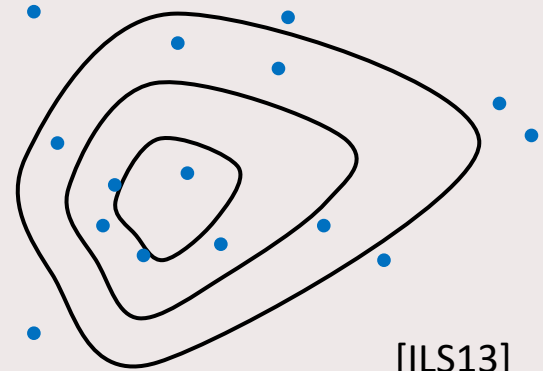
We sample a particle ensemble of size J from a prior distribution π_0 and update their positions as follows:

For $n = 0, 1, \dots$

i) Compute the model solution for each particle $\mu_n^{(j)}$:

$$u_n^{(j)} \in \mathcal{X} \quad \text{such that} \quad \left(\mathcal{M}_{\mu_n^{(j)}} u_n^{(j)}, \psi \right) = 0 \quad \forall \psi \in \mathcal{Y}$$

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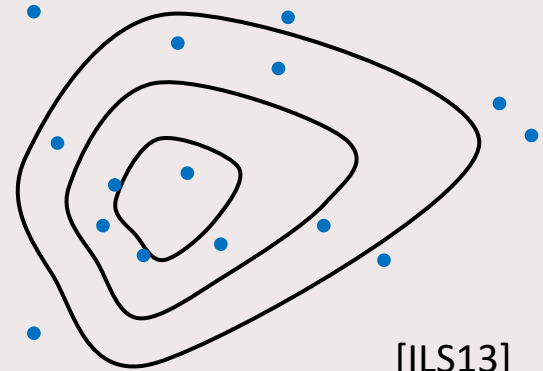
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ii) Compute the correlation matrices :

$$P_n := \text{sum} \left(\mathbf{L} u_n^{(j)} \otimes \mathbf{L} u_n^{(j)} - \mathbf{L} \bar{u}_n \otimes \mathbf{L} \bar{u}_n \right) \cdot (J - 1)^{-1}$$

$$Q_n := \text{sum} \left(\boldsymbol{\mu}_n^{(j)} \otimes \mathbf{L} u_n^{(j)} - \bar{\boldsymbol{\mu}}_n \otimes \mathbf{L} \bar{u}_n \right) \cdot (J - 1)^{-1}$$

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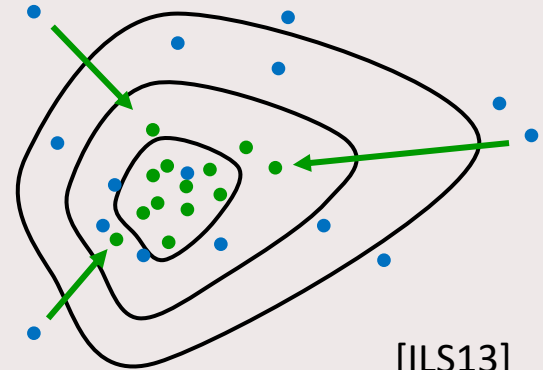
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For $n = 0, 1, \dots$

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$$\boldsymbol{\mu}_{n+1}^{(j)} = \boldsymbol{\mu}_n^{(j)} + Q_n(\Sigma + P_n)^{-1} (\mathbf{y} - \mathbf{L}u_n^{(j)})$$

- $\boldsymbol{\mu}_0^{(j)} \sim \pi_0 := e^{-\mathcal{J}(\boldsymbol{\mu})}$
- $\boldsymbol{\mu}_{n+1}^{(j)} \sim \pi_0 \cdot (e^{-\mathcal{J}(\boldsymbol{\mu}|\mathbf{y})})^{n+1}$



[ILS13]

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PARABOLIC pPDEs : SPACE-TIME CONSTRAINT

$$\begin{aligned} \partial_t u(\mathbf{x}, t; \boldsymbol{\mu}) + \mathcal{F}_{\boldsymbol{\mu}} u(\mathbf{x}, t; \boldsymbol{\mu}) &= 0 && \text{for any } \mathbf{x} \in \Omega \subset \mathbb{R}^d \text{ and } t \in I := [0, T] \\ u(\mathbf{x}, 0; \boldsymbol{\mu}) - u_0(\mathbf{x}, \boldsymbol{\mu}) &= 0 && \text{for any } \mathbf{x} \in \Omega \subset \mathbb{R}^d \end{aligned}$$

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to which corresponds the variational formulation:

$$\begin{aligned} \int_I \langle \partial_t u(\mathbf{x}, t; \boldsymbol{\mu}) + \mathcal{F}_{\boldsymbol{\mu}} u(\mathbf{x}, t; \boldsymbol{\mu}), v(\mathbf{x}, t) \rangle_{\mathcal{H}} dt &= 0 && \forall v(\mathbf{x}, t) \in L^2(I, \mathcal{V}) \\ \langle u(\mathbf{x}, 0; \boldsymbol{\mu}) - u_0(\mathbf{x}, \boldsymbol{\mu}), \xi(\mathbf{x}) \rangle_{\mathcal{H}} &= 0 && \forall \xi(\mathbf{x}) \in \mathcal{H} \end{aligned}$$

PARABOLIC pPDEs : SPACE-TIME CONSTRAINT

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ψ y

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ψ

\mathcal{Y}

that can be written as:

$$(\mathcal{M}_{\boldsymbol{\mu}} u, \psi)_{\mathcal{Y}} = 0 \quad \forall \psi \in \mathcal{Y}$$

SPACE-TIME WEAK MODEL

[UP14]

NUMERICAL APPROXIMATION

the infinite dimensional problem can be approximated by Petrov-Galerkin projection

$$\text{find } u_\varepsilon \in \mathcal{X}_\varepsilon \subset \mathcal{X} \quad \text{such that} \quad (\mathcal{M}_\mu u_\varepsilon, \psi_i) = 0 \quad \text{for all } \psi_i \in \mathcal{Y}_\varepsilon \subset \mathcal{Y}$$

NUMERICAL APPROXIMATION

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where

\mathcal{X}_ε : **trial space** ← must ensure good approximation

\mathcal{Y}_ε : **test space** ← must ensure proper stability

NUMERICAL APPROXIMATION : REDUCED BASIS METHODS

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Reduced Basis (RB) methods employ a set of pre-computed solutions to choose an optimal couple $(\mathcal{X}_\varepsilon, \mathcal{Y}_\varepsilon)$.

[BHL93]

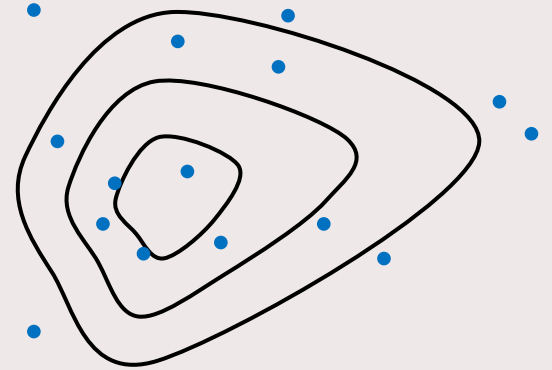
[HO08]

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THE (REDUCED BASIS) ENSEMBLE KALMAN METHOD

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[ILS13]

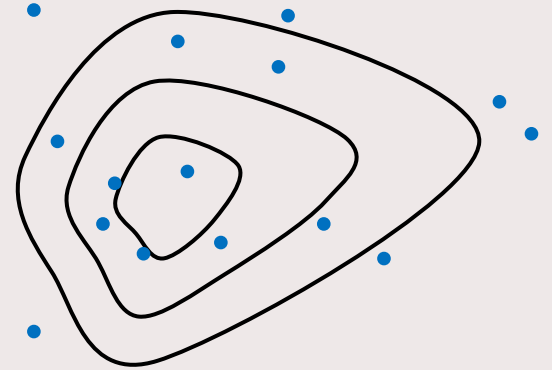
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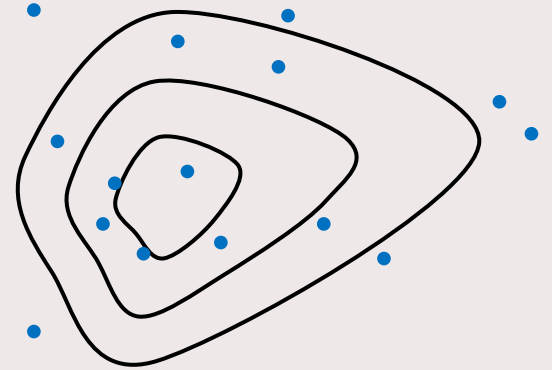
THE REDUCED BASIS ENSEMBLE KALMAN METHOD

We sample a particle ensemble of size J from a prior distribution π_0 and update their positions as follows:

For $n = 0, 1, \dots$

i) Compute the model solution for each particle $\boldsymbol{\mu}_n^{(j)}$:

$$u_{\varepsilon,n}^{(j)} \in \mathcal{X}_\varepsilon \quad \text{such that} \quad \left(\mathcal{M}_{\boldsymbol{\mu}_n^{(j)}} u_{\varepsilon,n}^{(j)}, \psi_i \right) = 0 \quad \forall \psi_i \in \mathcal{Y}_\varepsilon$$



THE ENSEMBLE KALMAN METHOD

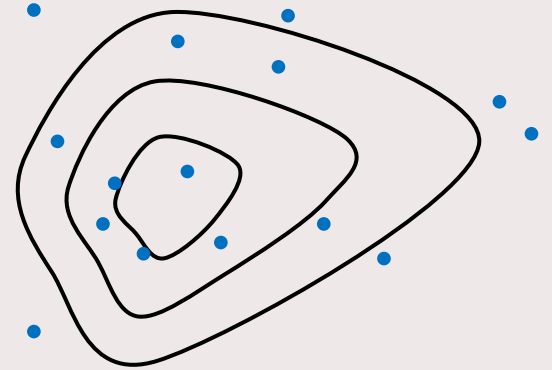
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$$Q_n := \text{sum} \left(\boldsymbol{\mu}_n^{(j)} \otimes \mathbf{L} u_n^{(j)} - \bar{\boldsymbol{\mu}}_n \otimes \mathbf{L} \bar{u}_n \right) \cdot (J - 1)^{-1}$$



THE REDUCED BASIS ENSEMBLE KALMAN METHOD

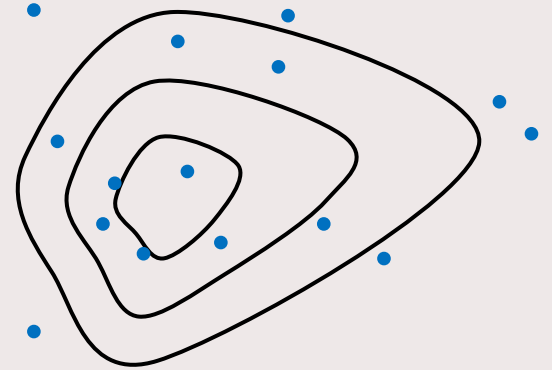
We sample a particle ensemble of size J from a prior distribution π_0 and update their positions as follows:

For $n = 0, 1, \dots$

ii) Compute the correlation matrices :

$$P_{\epsilon,n} := \text{sum} \left(\mathbf{L} u_{\epsilon,n}^{(j)} \otimes \mathbf{L} u_{\epsilon,n}^{(j)} - \mathbf{L} \bar{u}_{\epsilon,n} \otimes \mathbf{L} \bar{u}_{\epsilon,n} \right) \cdot (J - 1)^{-1}$$

$$Q_{\epsilon,n} := \text{sum} \left(\boldsymbol{\mu}_n^{(j)} \otimes \mathbf{L} u_{\epsilon,n}^{(j)} - \bar{\boldsymbol{\mu}}_n \otimes \mathbf{L} \bar{u}_{\epsilon,n} \right) \cdot (J - 1)^{-1}$$



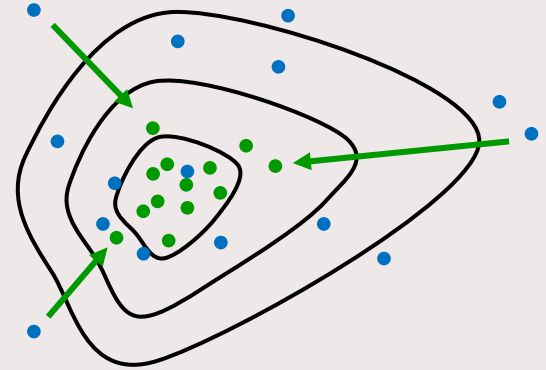
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$$\mu_{n+1}^{(j)} = \mu_n^{(j)} + Q_n(\Sigma + P_n)^{-1} (\mathbf{y} - \mathbf{L}u_n^{(j)})$$



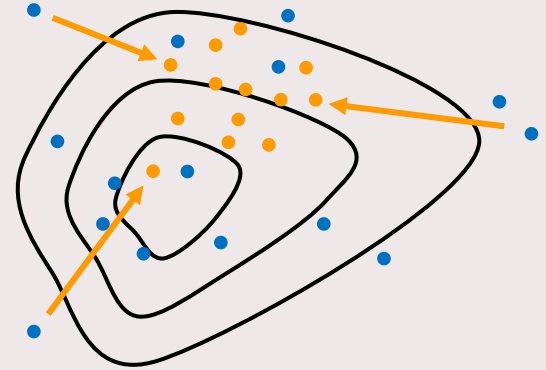
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$$\mu_{n+1}^{(j)} \stackrel{?}{=} \mu_n^{(j)} + Q_{\varepsilon,n} (\Sigma + P_{\varepsilon,n})^{-1} (\mathbf{y} - \mathbf{L} u_{\varepsilon,n}^{(j)})$$



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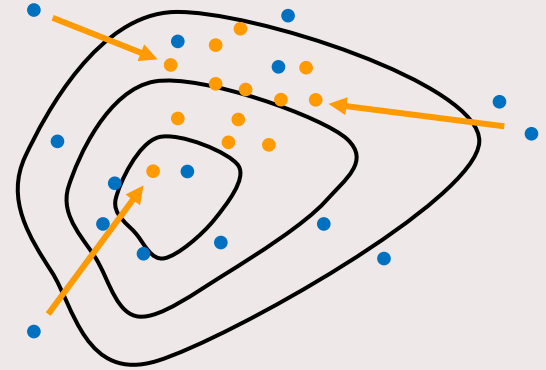
For $n = 0, 1, \dots$

iii) Update each particle $\boldsymbol{\mu}_n^{(j)}$ in the ensemble:

$$\boldsymbol{\mu}_{n+1}^{(j)} \neq \boldsymbol{\mu}_n^{(j)} + Q_{\varepsilon,n} (\boldsymbol{\Sigma} + P_{\varepsilon,n})^{-1} (\mathbf{y} - \mathbf{L} u_{\varepsilon,n}^{(j)})$$

Such an iteration would not converge to $\boldsymbol{\mu}_{\text{OPT}}$ because

$$\min_{\boldsymbol{\mu} \in \mathcal{P}} \frac{1}{2} \|\mathbf{y} - \mathbf{L} u\|_{\boldsymbol{\Sigma}^{-1}}^2 \neq \min_{\boldsymbol{\mu} \in \mathcal{P}} \frac{1}{2} \|\mathbf{y} - \mathbf{L} u_{\varepsilon}\|_{\boldsymbol{\Sigma}^{-1}}^2$$



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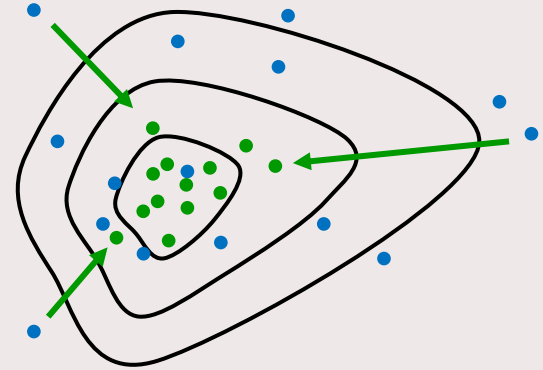
iii) Update each particle $\boldsymbol{\mu}_n^{(j)}$ in the ensemble:

$$\boldsymbol{\mu}_{n+1}^{(j)} = \boldsymbol{\mu}_n^{(j)} + Q_{\varepsilon,n} (\boldsymbol{\Sigma} + \boldsymbol{\Gamma}_{\varepsilon,n} + P_{\varepsilon,n})^{-1} (\mathbf{y} - \boldsymbol{\delta}_{\varepsilon,n} - \mathbf{L} u_{\varepsilon,n}^{(j)})$$

where

$$\boldsymbol{\delta}_{\varepsilon,n} := \frac{1}{J} \cdot \text{sum} \left(\mathbf{L} (u_{\varepsilon,n}^{(j)} - u_n^{(j)}) \right)$$

$$\boldsymbol{\Gamma}_{\varepsilon,n} := \frac{1}{J-1} \cdot \text{sum} \left(\mathbf{L} (u_{\varepsilon,n}^{(j)} - u_n^{(j)}) \otimes \mathbf{L} (u_{\varepsilon,n}^{(j)} - u_n^{(j)}) - \boldsymbol{\delta}_{\varepsilon,n} \otimes \boldsymbol{\delta}_{\varepsilon,n} \right)$$



[PMQ16]

[Cal+18]

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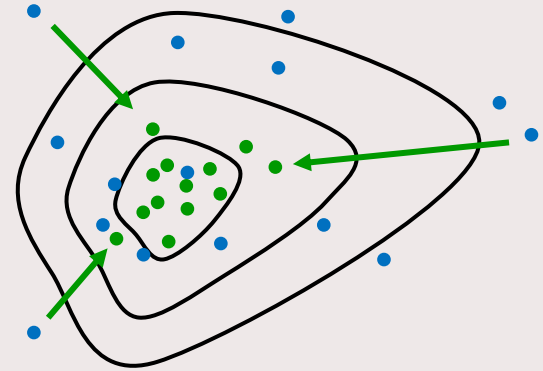
iii) Update each particle $\mu_n^{(j)}$ in the ensemble:

$$\mu_{n+1}^{(j)} \approx \mu_n^{(j)} + Q_{\varepsilon,n} (\Sigma + \Gamma_{\varepsilon,0} + P_{\varepsilon,n})^{-1} (\mathbf{y} - \delta_{\varepsilon,0} - \mathbf{L} u_{\varepsilon,n}^{(j)})$$

where

$$\delta_{\varepsilon,0} := \frac{1}{J} \cdot \text{sum} \left(\mathbf{L} (u_{\varepsilon,0}^{(j)} - u_0^{(j)}) \right)$$

$$\Gamma_{\varepsilon,0} := \frac{1}{J-1} \cdot \text{sum} \left(\mathbf{L} (u_{\varepsilon,0}^{(j)} - u_0^{(j)}) \otimes \mathbf{L} (u_{\varepsilon,0}^{(j)} - u_0^{(j)}) - \delta_{\varepsilon,0} \otimes \delta_{\varepsilon,0} \right)$$



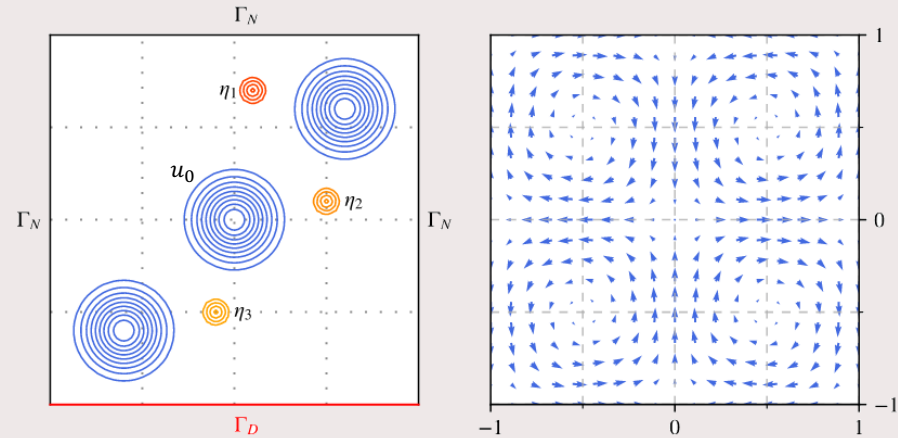
← same $u_0^{(j)}$ used for training the RB model

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ADVECTION-DISPERSION PROBLEM

$$\frac{\partial u}{\partial t} - \overset{\text{unknown}}{\mu} \cdot \Delta u(t) + \mathbf{v} \cdot \nabla u(t) = 0 \quad \text{on } \Omega := (-1, +1)^2 \quad \text{with} \quad \mathbf{v} = \begin{bmatrix} +\sin(\pi x_1)\cos(\pi x_2) \\ -\cos(\pi x_1)\sin(\pi x_2) \end{bmatrix}$$
$$u(0) = u_0$$



[Kär+18]

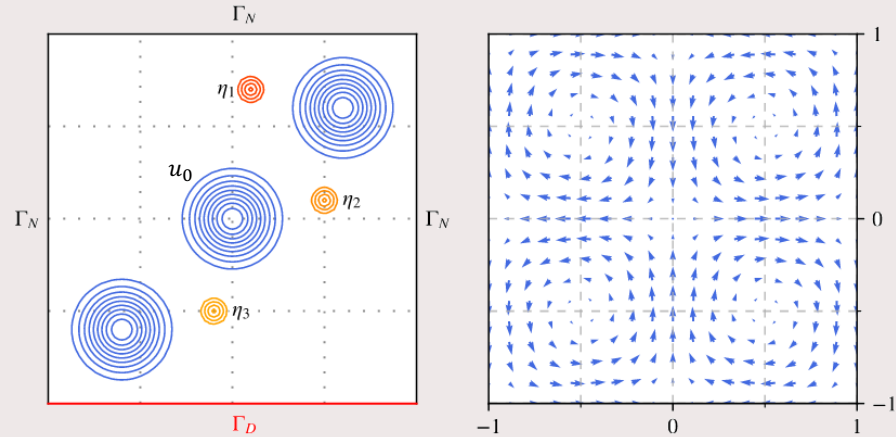
ADVECTION-DISPERSION PROBLEM

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$$u(0) = u_0$$

we consider:

- 3 sensor locations
- 40 time-activations per sensor
- $t \in (0, 2.4)$
- $\boldsymbol{\mu} \in [1/50, 1/10]$



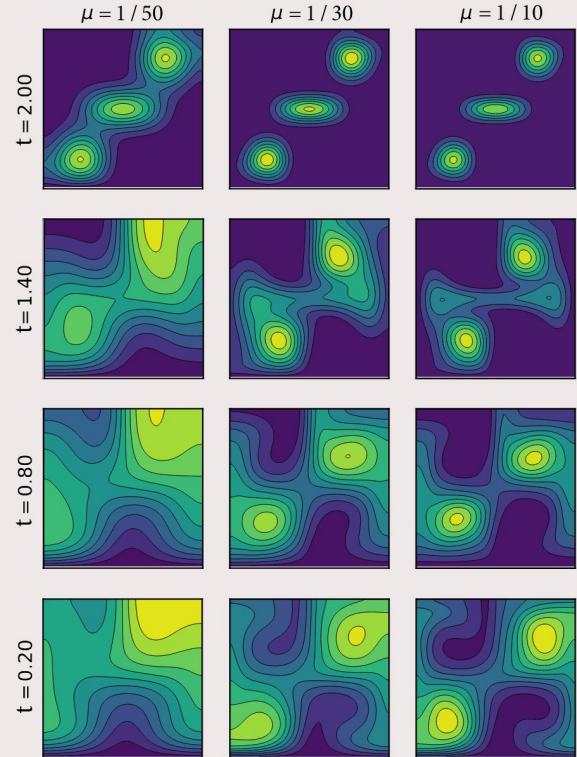
MODEL ORDER REDUCTION

considering a fine **FE discretization** as exact model

FE dofs spatial discretization = 10100 (P2-P2 G)

FE dofs time discretization = 240 (P1-P0 PG)

[Hec12]



MODEL ORDER REDUCTION

[Gre12]

considering a fine **FE discretization** as exact model

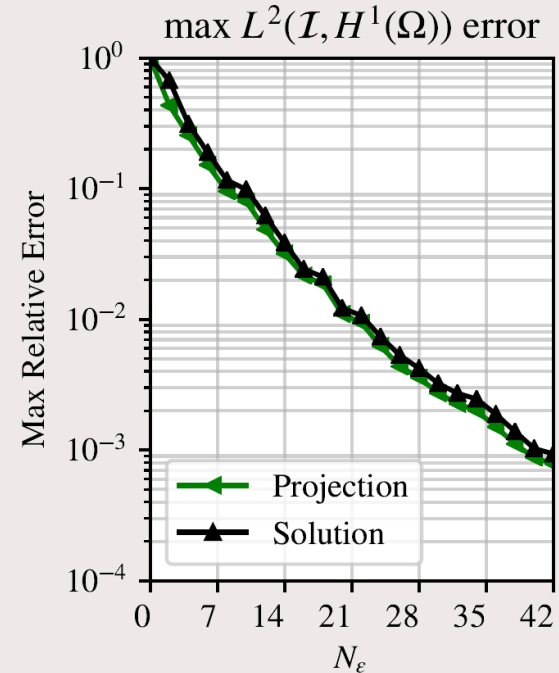
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employing the **weak-greedy-POD** algorithm, we achieve relative error $\varepsilon < 10^{-3}$ with 42 spatial basis functions

RB dofs spatial discretization = N_ε (RB-RB G)

FE dofs time discretization = 240 (P1-P0 PG)



MODEL ORDER REDUCTION

considering a fine **FE discretization** as exact model

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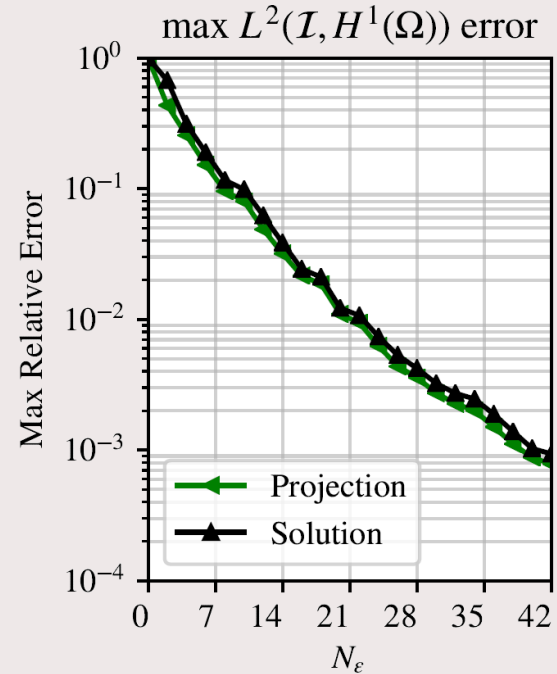
FE dofs time discretization = 240 (P1-P0 PG)

employing the **weak-greedy-POD** algorithm, we achieve relative error $\varepsilon < 10^{-3}$ with 42 spatial basis functions

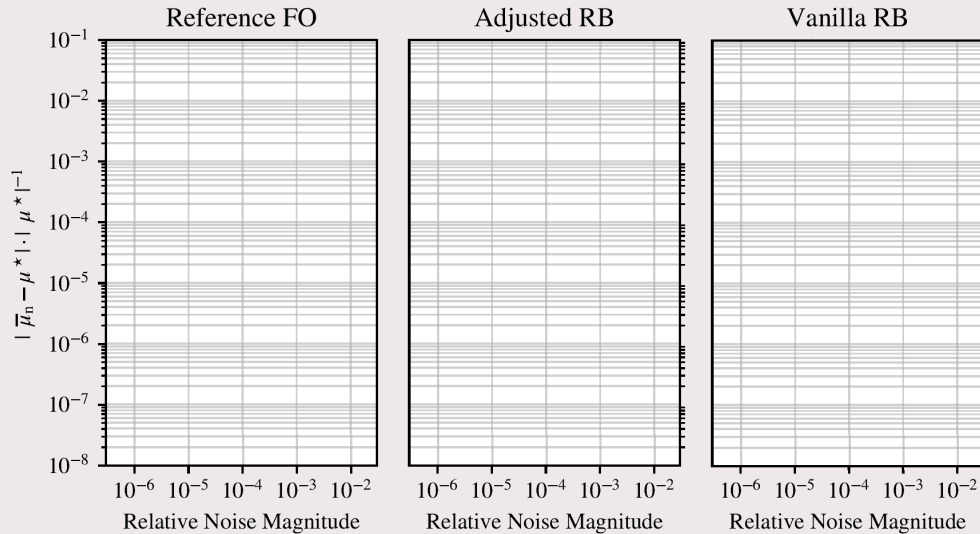
RB dofs spatial discretization = N_ε (RB-RB G)

FE dofs time discretization = 240 (P1-P0 PG)

training time ~2 min, speed up $\times 250$



PARAMETER ESTIMATION : NOISE EFFECTS



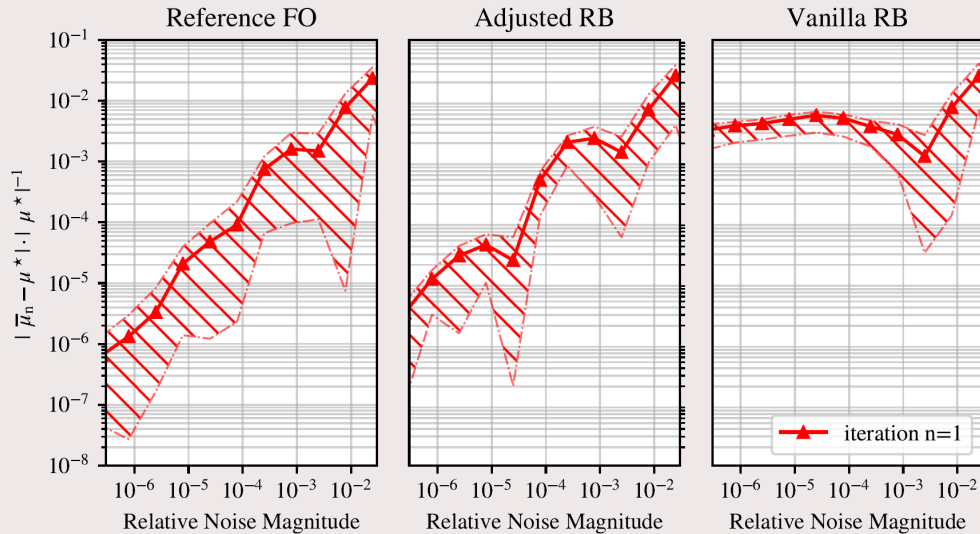
we try to estimate the $\mu^* = 1/25$
from noisy observations of $u(\mu^*)$

we consider different relative noise
magnitudes $\lambda_{\max}^{\frac{1}{2}}(\Sigma)/\|\mathbf{L}u(\mu^*)\|_{\infty}$

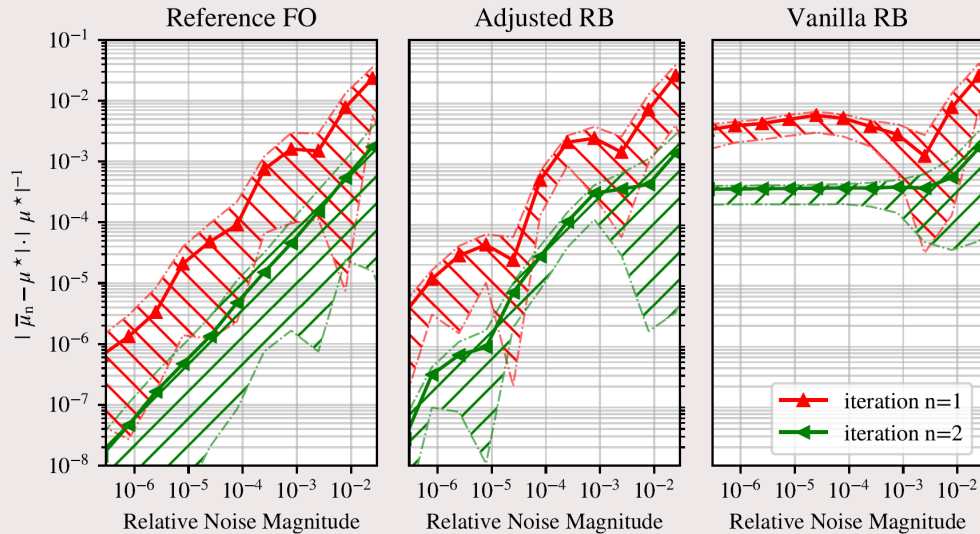
we sample ensembles of size $J = 40$
from the prior $\pi_0 = U(1/10, 1/50)$

we replicate the analysis 64 times
for each noise level

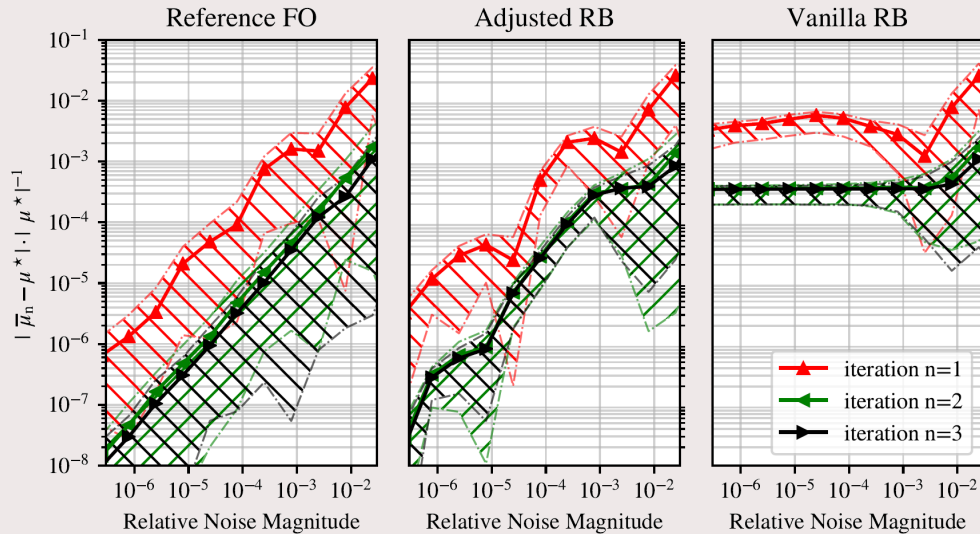
PARAMETER ESTIMATION : NOISE EFFECTS



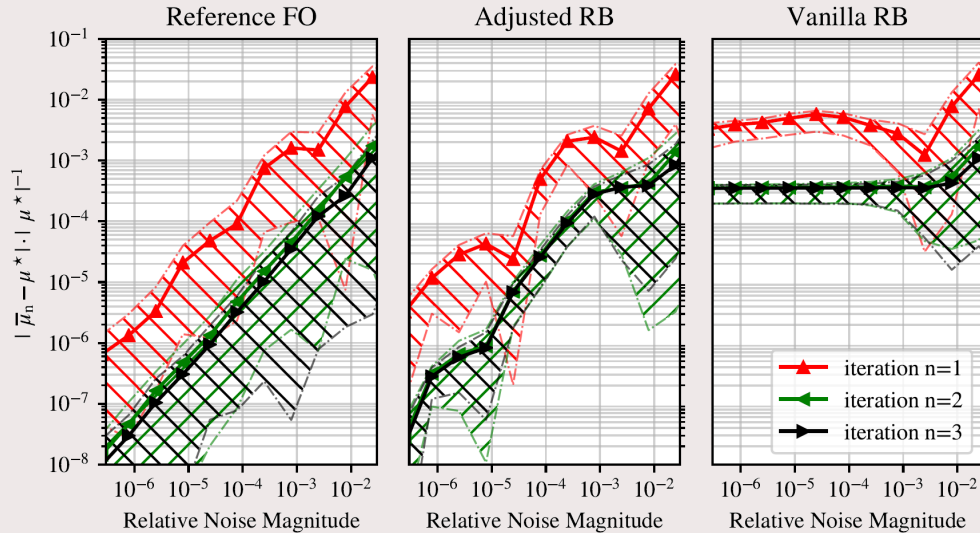
PARAMETER ESTIMATION : NOISE EFFECTS



PARAMETER ESTIMATION : NOISE EFFECTS



PARAMETER ESTIMATION : NOISE EFFECTS



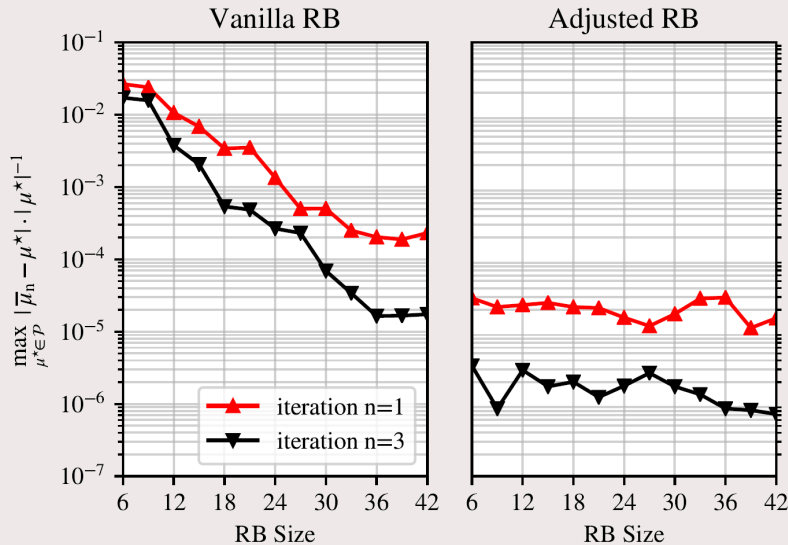
results show a linear convergence when the exact FO model is employed

the error stagnates when the model bias is not corrected in the RB-EnKM

the adjusted RB-EnKM shows an error decay comparable with the FO one

the cost of the RB-EnKM is just $\sim 4\%$ of the cost of the standard EnKM

PARAMETER ESTIMATION : REDUCED BASIS SIZE



when the measurements bias is not corrected, the relative error is strictly dependent on the RB model accuracy

with the bias correction, the performances of the method are made independent on the RB size (at least for this problem)

OUTLINE

1. INTRODUCTION
2. VARIATIONAL METHODS
3. MODEL APPROXIMATION
4. THE REDUCED BASIS ENSEMBLE KALMAN METHOD
5. NUMERICAL EXPERIMENTS
6. CONCLUSIONS

CONCLUSIONS

SUMMARY :

- we introduced Reduced Basis solvers to improve the EnKM efficiency
- we adjusted the method to guarantee the robustness to model-biases
- we tested the method both on linear and non-linear 2D problems

OUTLOOK :

- the bias correction could be updated as the particles distribution evolves
- the approach could be extended to synchronous data assimilation problems

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THANKS FOR YOUR ATTENTION!

QUESTION TIME