A multi-fidelity ensemble Kalman filter with adaptive reduced-order models

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OUTLINE

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Cecilia Pagliantini : Department of Mathematics, University of Pisa

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ACKNOWLEDGMENTS

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DATA ASSIMILATION

DYNAMICAL MODEL

$$\omega_{\mathrm{n+1|n}} = \mathcal{M}\,\omega_{\mathrm{n|n}}$$

DATA ASSIMILATION

EXPERIMENTAL MEASUREMENTS

$$\boldsymbol{y}_{\mathrm{n+1}} = \mathbf{H}\omega_{\mathrm{T}} + \boldsymbol{\epsilon}_{\mathrm{n+1}}$$

DYNAMICAL MODEL

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DYNAMICAL MODEL

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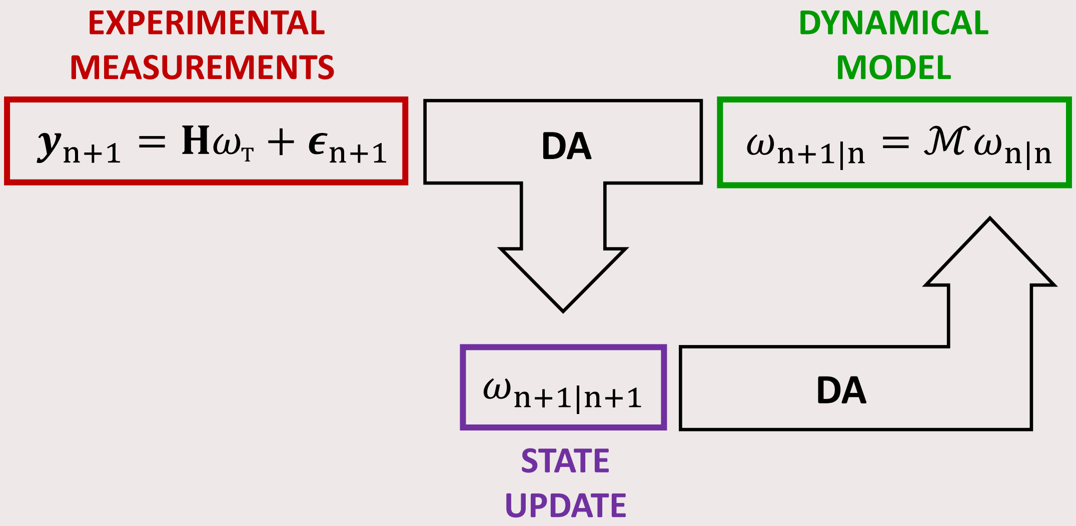
DA

$$\omega_{n+1|n+1}$$

STATE UPDATE

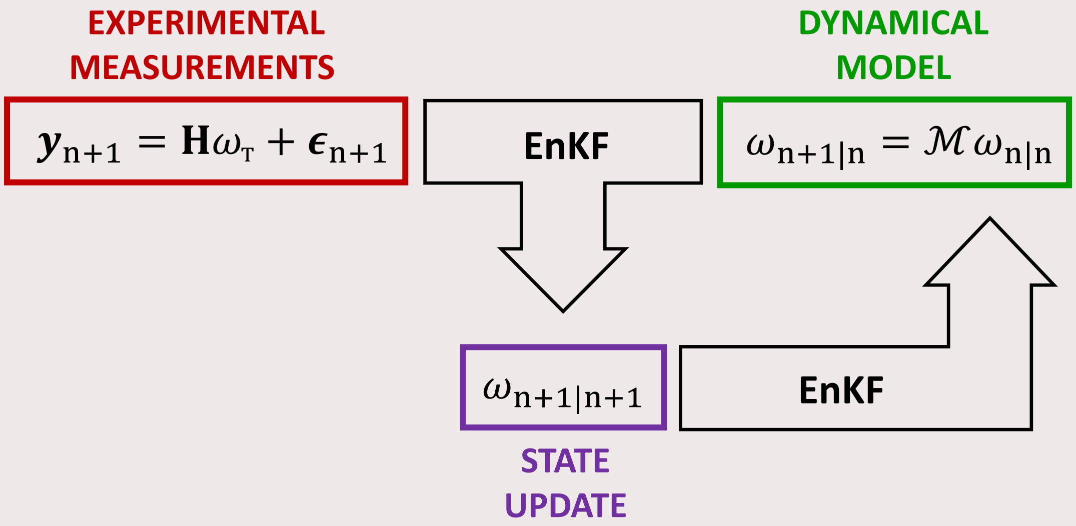
DATA ASSIMILATION



DATA ASSIMILATION

[Kal60]

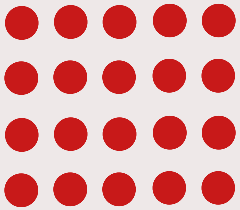
R. E. Kalman. "A new approach to linear filtering and prediction problems". (1960)



THE ENSEMBLE KALMAN FILTER

PREDICT : $\omega^{(j)}_{n+1|n} = \mathcal{M}\omega^{(j)}_{n|n}$

(j)



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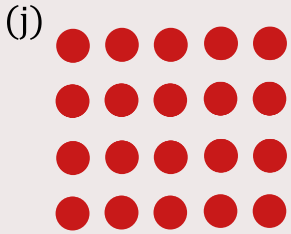
PREDICT : $\omega^{(j)}_{n+1|n} = \mathcal{M}\,\omega^{(j)}_{n|n}$

ESTIMATE : $\hat{\mathbf{C}}_{n+1|n} = \mathrm{cov}\left\{\omega^{(j)}_{n+1|n}\right\}$



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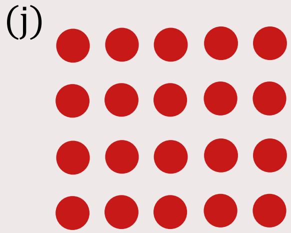
ANALYSE : $\omega^{(j)}_{n+1|n+1} = \omega^{(j)}_{n+1|n} + \widehat{\mathbf{K}}_{n+1}\left(\boldsymbol{y}_{n+1} - \mathbf{H}\omega^{(j)}_{n+1|n}\right)$



employing the empirical Kalman gain

$$\widehat{\mathbf{K}}_{n+1} = \hat{\mathbf{C}}_{n+1|n}\mathbf{H}^*(\mathbf{H}\hat{\mathbf{C}}_{n+1|n}\mathbf{H}^* + \boldsymbol{\Sigma})^{-1}$$

THE ENSEMBLE KALMAN FILTER

PREDICT : $\omega^{(j)}_{n+1|n} = \mathcal{M}\,\omega^{(j)}_{n|n}$ ← **EXPENSIVE!**

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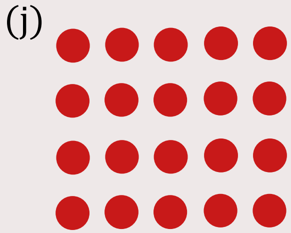

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[PMS21]

A. Popov, et all. "A multifidelity ensemble Kalman filter with reduced order control variates." (2021)

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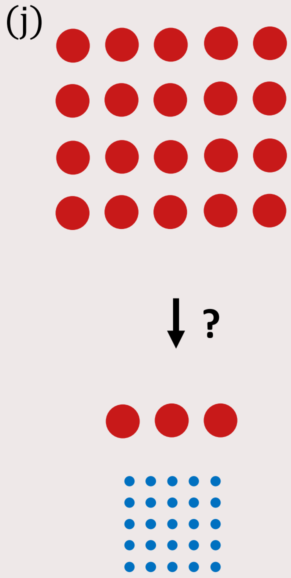
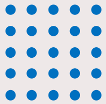


THE MULTI-FIDELITY ENSEMBLE KALMAN FILTER : SOME COMPLICATION

$\omega_{n|n}^{(j)}$

Principal Ensemble

Ancillary Ensemble



$\widetilde{\omega}_{n|n,\varepsilon}^{(j)}$

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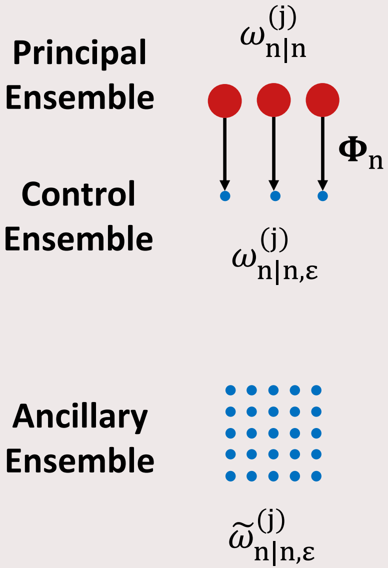

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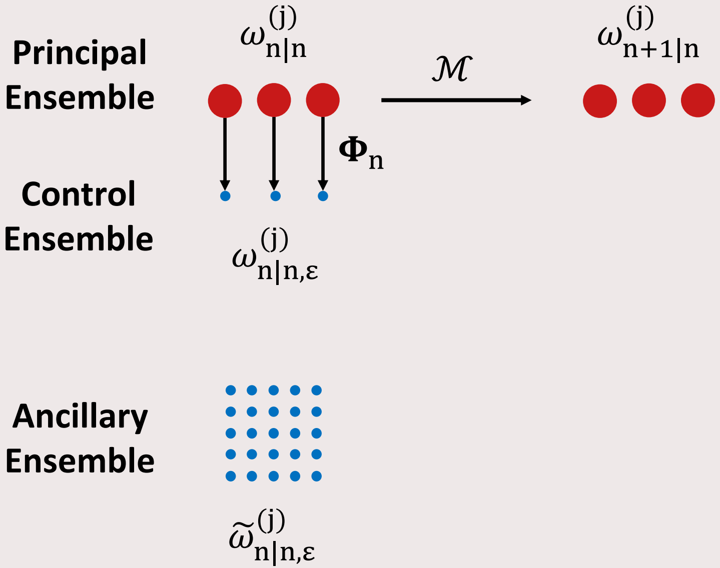

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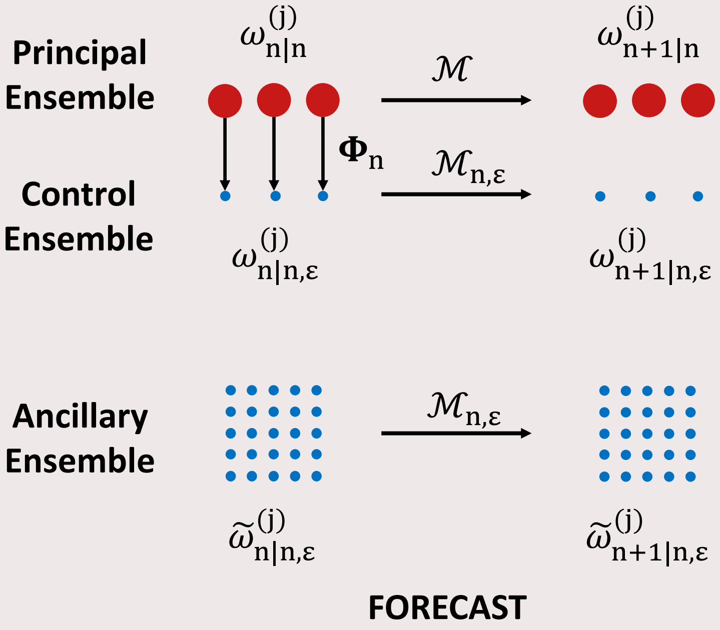

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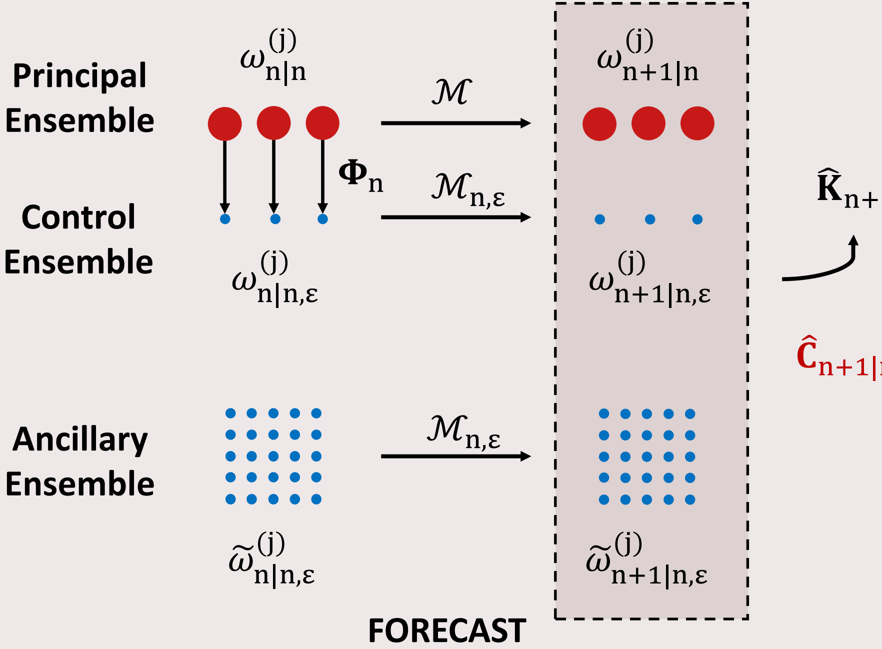

$$\widehat{\mathbf{K}}_{n+1} = \widehat{\mathbf{C}}_{n+1|n}\mathbf{H}^*(\mathbf{H}\widehat{\mathbf{C}}_{n+1|n}\mathbf{H}^* + \boldsymbol{\Sigma})^{-1}$$

$$\begin{aligned}\widehat{\mathbf{C}}_{n+1|n} = {} & \mathrm{cov}\left(\omega^{(j)}_{n+1|n}, \omega^{(j)}_{n+1|n}\right) + \\ & -\tfrac{1}{2}\,\mathrm{cov}\left(\boldsymbol{\Phi}_n^*\omega^{(j)}_{n+1|n,\varepsilon}, \omega^{(j)}_{n+1|n}\right) \\ & -\tfrac{1}{2}\,\mathrm{cov}\left(\omega^{(j)}_{n+1|n}, \boldsymbol{\Phi}_n^*\widetilde{\omega}^{(j)}_{n+1|n,\varepsilon}\right) \\ & +\tfrac{1}{4}\,\mathrm{cov}\left(\boldsymbol{\Phi}_n^*\omega^{(j)}_{n+1|n,\varepsilon}, \boldsymbol{\Phi}_n^*\omega^{(j)}_{n+1|n,\varepsilon}\right) \\ & +\tfrac{1}{4}\,\mathrm{cov}\left(\boldsymbol{\Phi}_n^*\widetilde{\omega}^{(j)}_{n+1|n,\varepsilon}, \boldsymbol{\Phi}_n^*\widetilde{\omega}^{(j)}_{n+1|n,\varepsilon}\right)\end{aligned}$$

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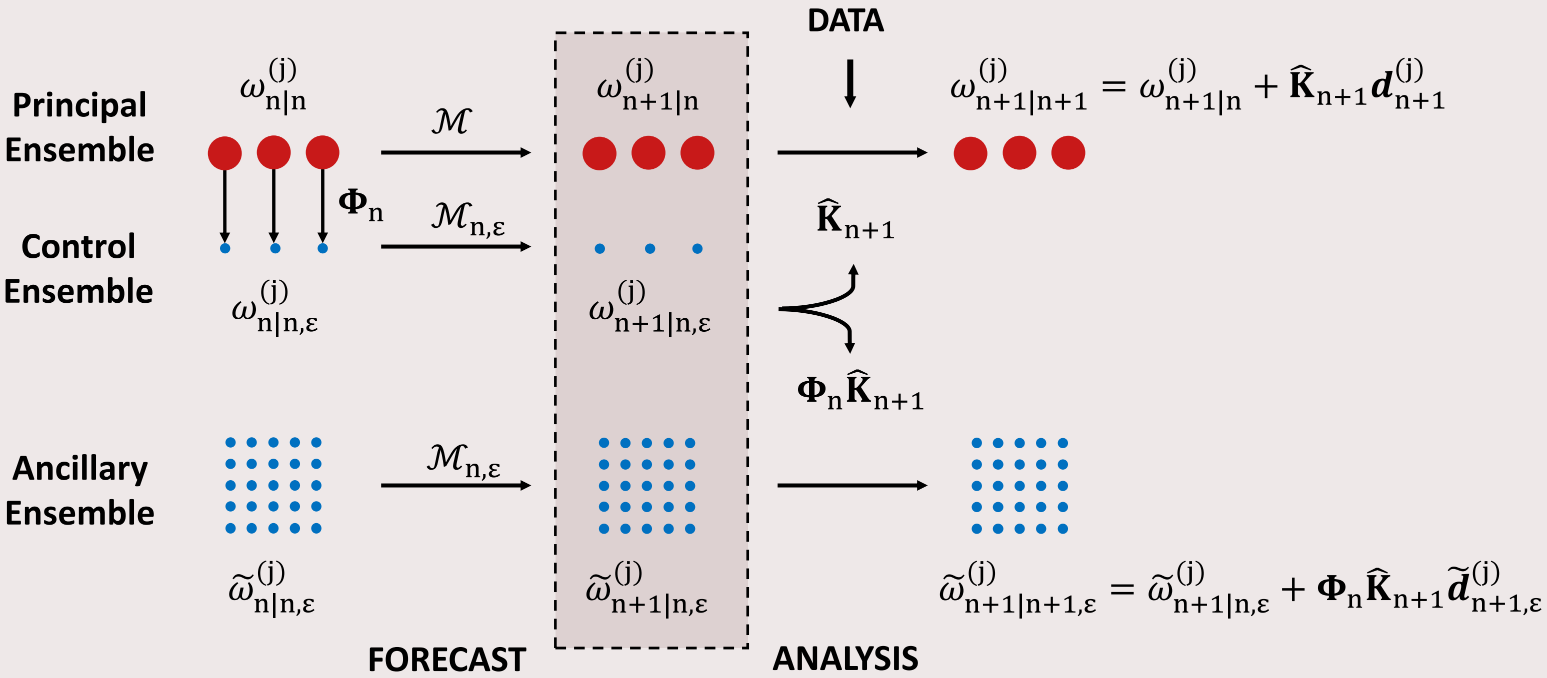

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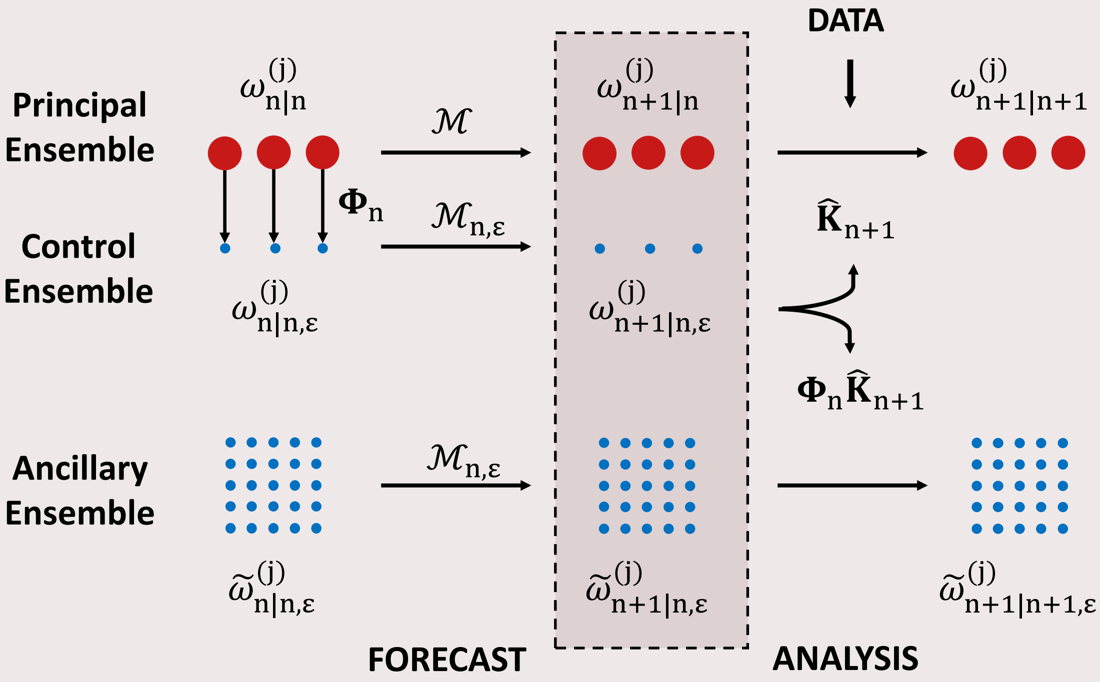
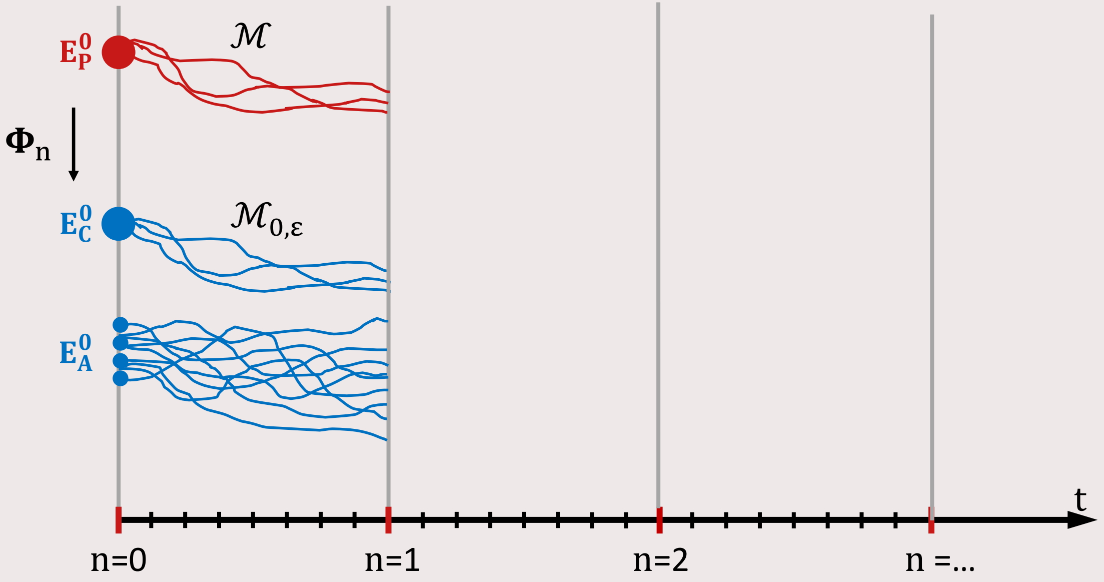


? $\mathcal{M}_{n,\varepsilon}$?

[DM22]

G. Donoghue and M. Yano. "A multi-fidelity ensemble Kalman filter with hyperreduced reduced-order models". (2022)

THE MULTI-FIDELITY ENSEMBLE KALMAN FILTER



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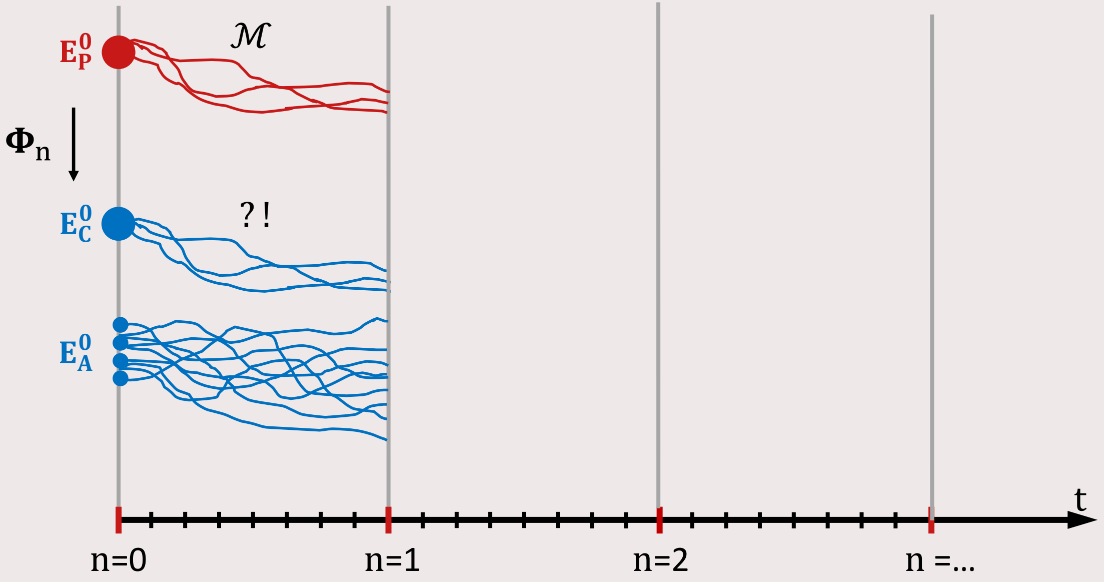
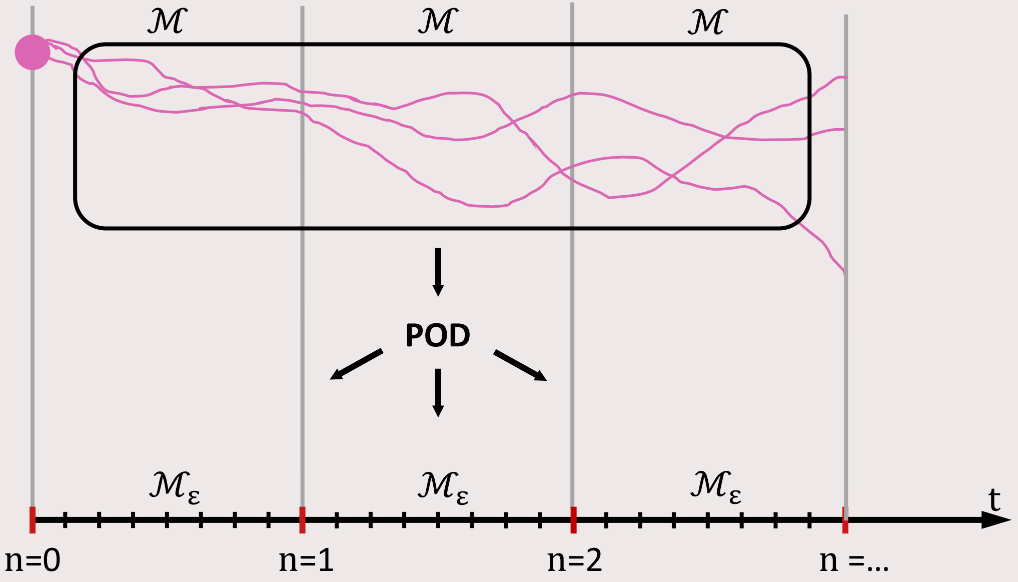


RB MODEL CONSTRUCTION : POPOV'S APPROACH



a few long trajectories are used to build offline a global RB model

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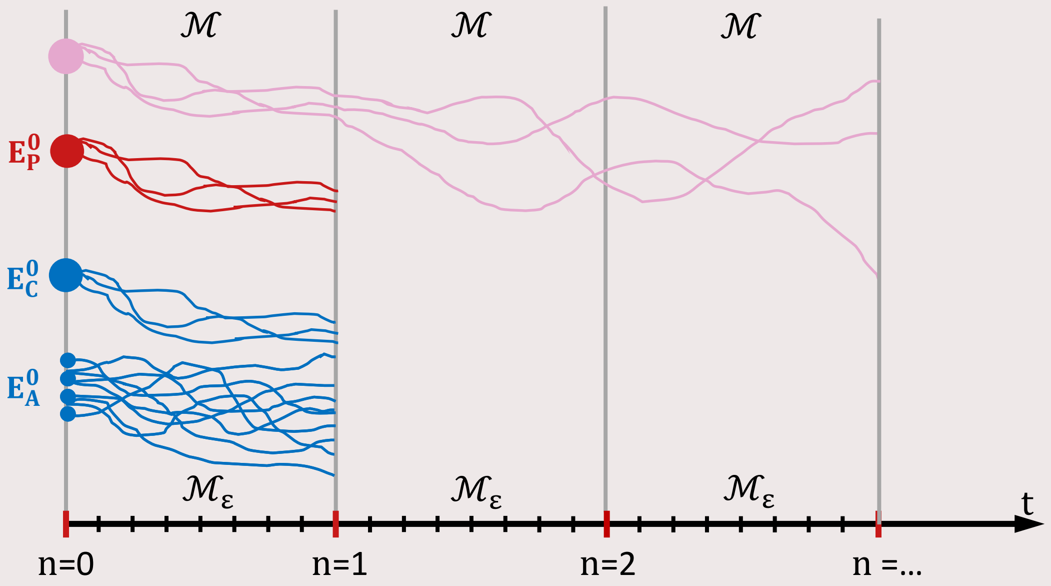

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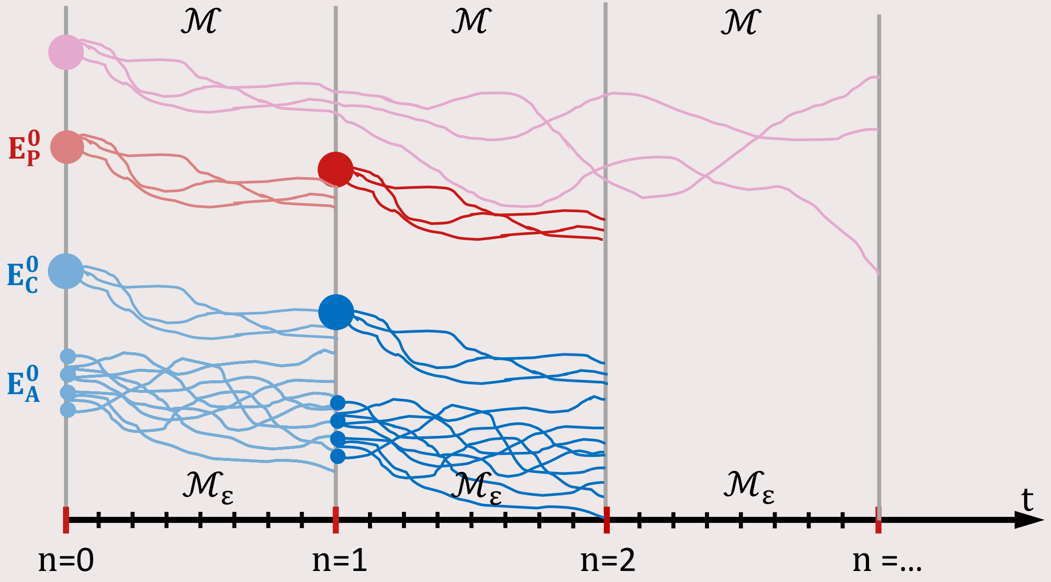

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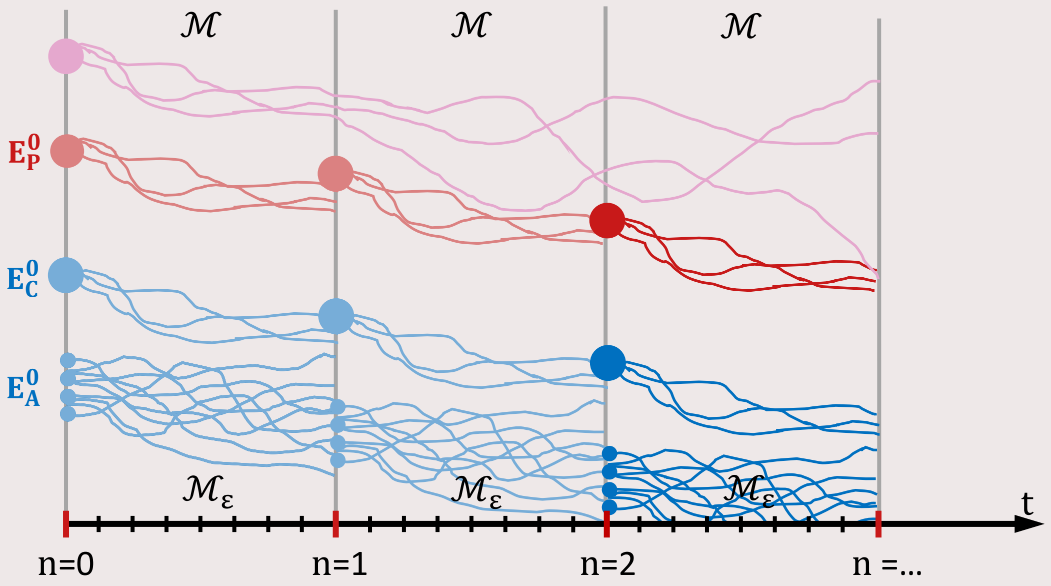


a few long trajectories are used to build offline a global RB model

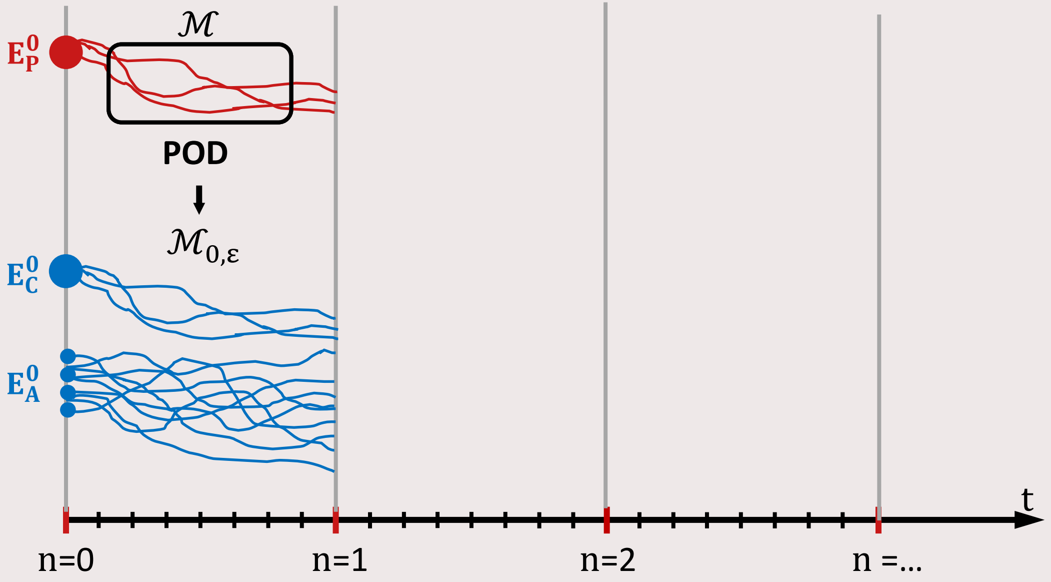
PROs:

- easy to implement
- can incorporate steady states

CONs:

- suffers initial uncertainty
- leads to large RB models
- might introduce biases

RB MODEL CONSTRUCTION : DONOGHYE'S APPROACH



the principal trajectories are used to build RB models on-the-fly

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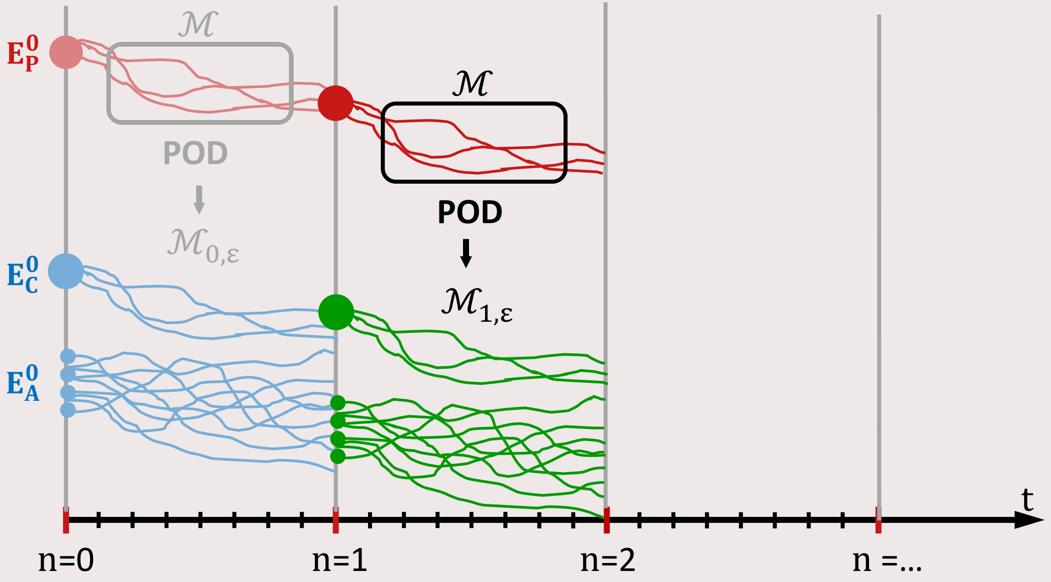

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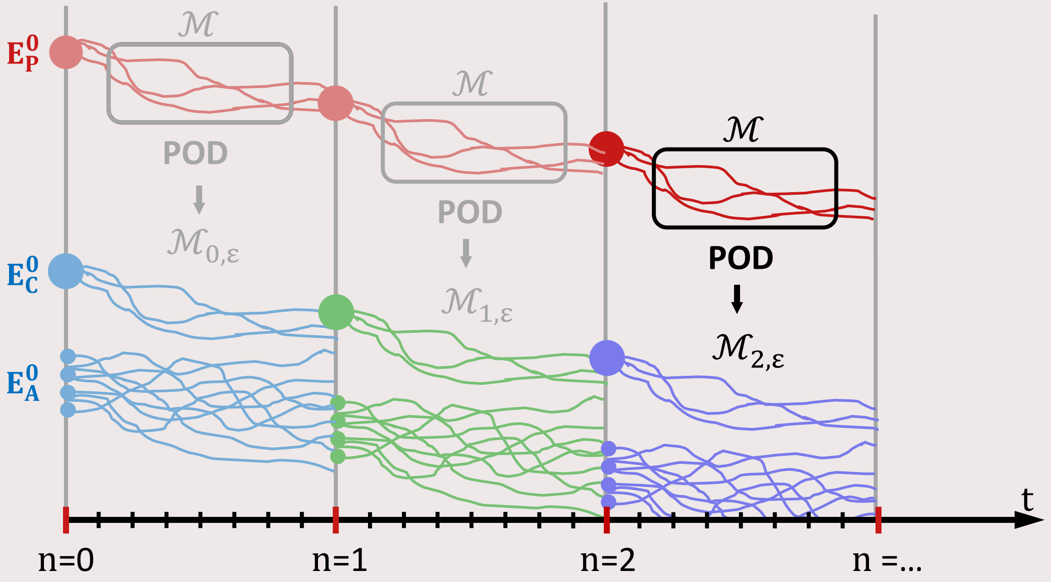


the principal trajectories are used to build RB models on-the-fly

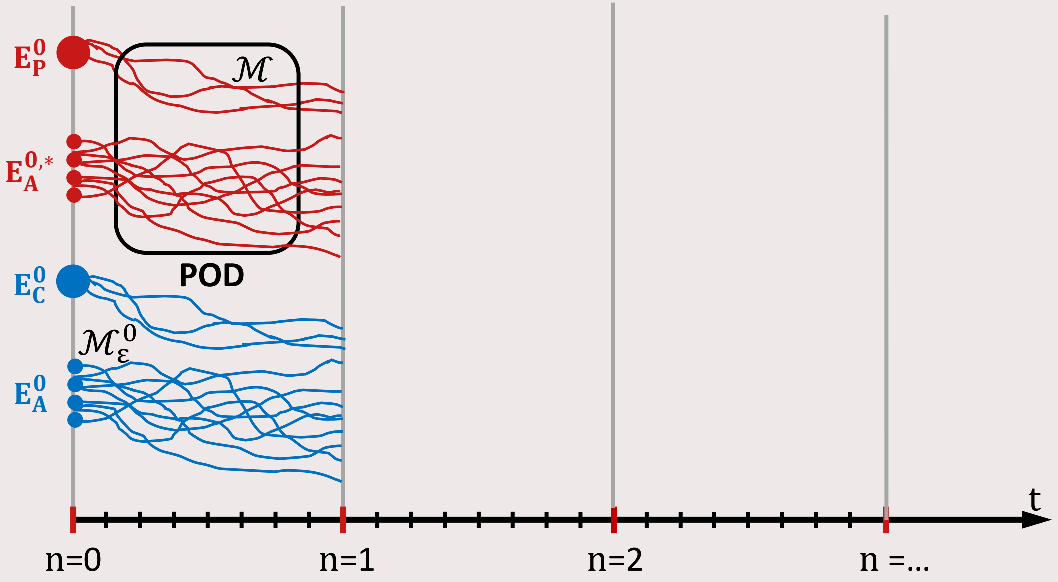
PROs:

- leads to small RB models
- doesn't introduce biases

CONs:

- requires constant retraining
- too little information extracted (poor accuracy for the RB model)

RB MODEL CONSTRUCTION : OUR APPROACH



principal and auxiliary ancillary trajectories are used to build RB models on-the-fly

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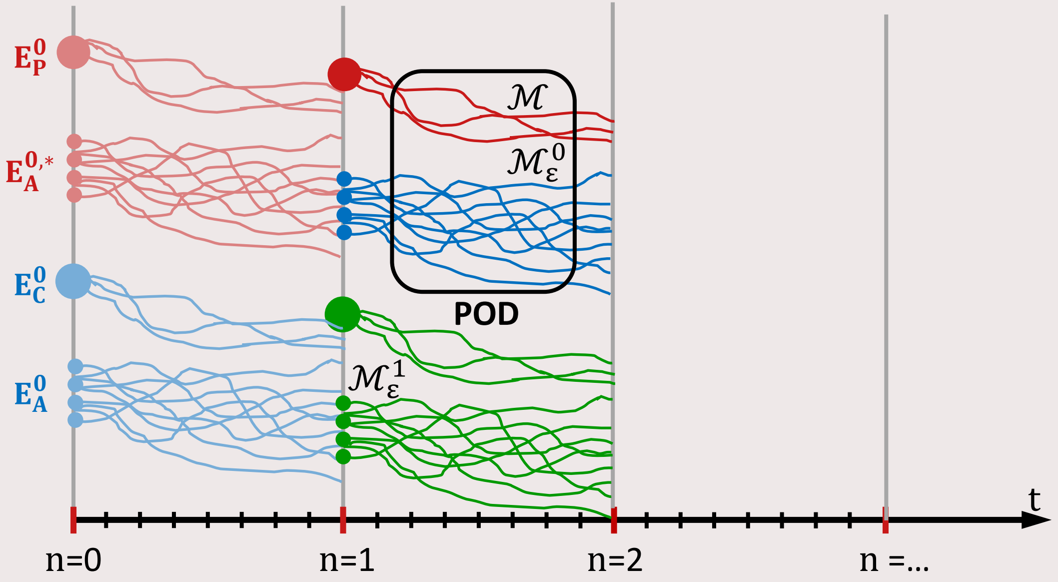

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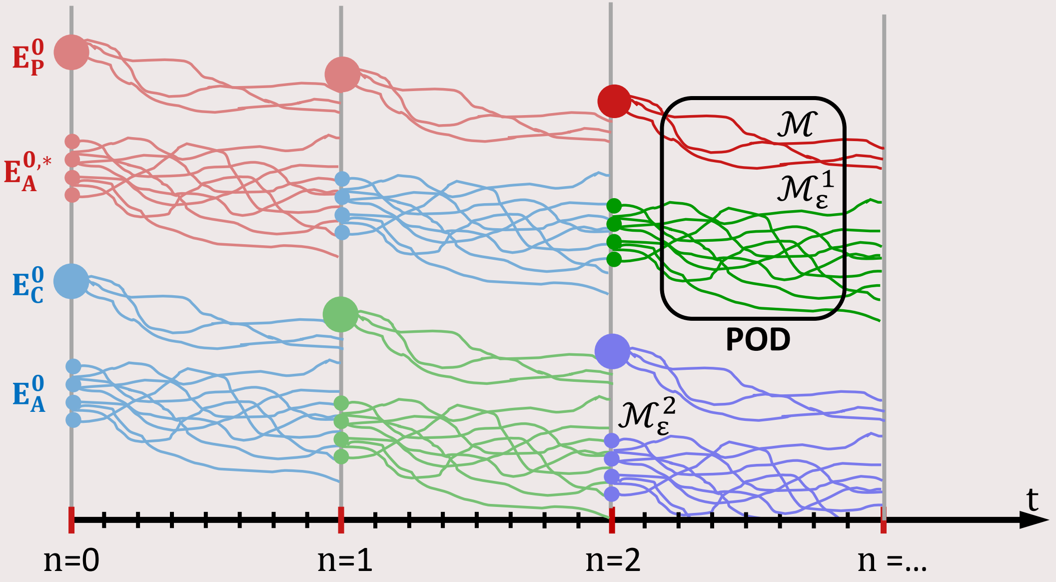


principal and auxiliary ancillary trajectories are used to build RB models on-the-fly

PROs:

- retains past model's information
- achieves good accuracy

CONs:

- first step is significantly expensive
- large range of RB model sizes

QUASI-GEOSTROPHIC EQUATIONS

[MWW20]

C. Mou, Z. Wang, D. Wells, X. Xie and T. Iliescu. "Reduced order models for the quasi-geostrophic equations: A brief survey." (2020)

find $\omega = \omega(x, y, t)$, $\psi = \psi(x, y, t)$ such that

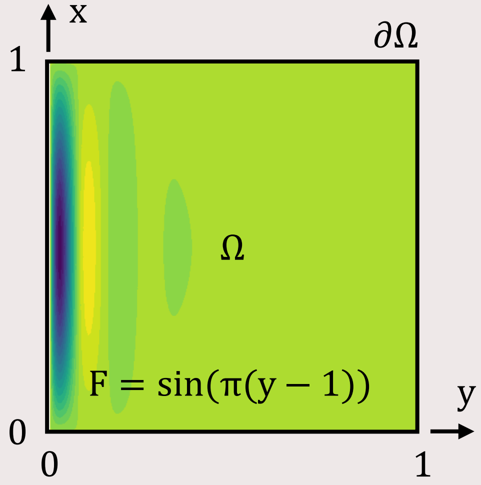
$$\partial_t \omega = \mathrm{Ro}\, J(\omega, \psi) + \partial_x \psi + \frac{\mathrm{Ro}}{\mathrm{Re}} \Delta\omega + F, 0\Delta\psi + \omega = 0 \quad \longleftarrow \quad J(\omega, \psi) = \partial_x \psi \partial_y \omega - \partial_x \psi \partial_y \omega$$

given the boundary and initial conditions

$$\omega(x, y, t) = 0, \qquad (x, y) \in \partial\Omega$$
$$\psi(x, y, t) = 0, \qquad (x, y) \in \partial\Omega$$
$$\omega(x, y, 0) = \omega_0, \qquad (x, y) \in \Omega$$

and

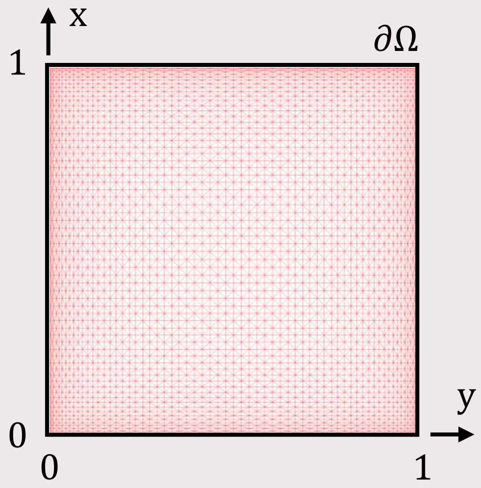
$$\partial_x \psi_0 + \frac{\mathrm{Ro}}{\mathrm{Re}} \Delta\omega_0 + F = 0, \;\; \Delta\psi_0 + \omega_0 = 0$$



QUASI-GEOSTROPHIC EQUATIONS

high-fidelity physical model constructed considering:

- fully implicit mid-point discretization in time $(\mathrm{dt} = 0.1)$
- P1 finite elements discretization in space $(4225\ \mathrm{dofs})$



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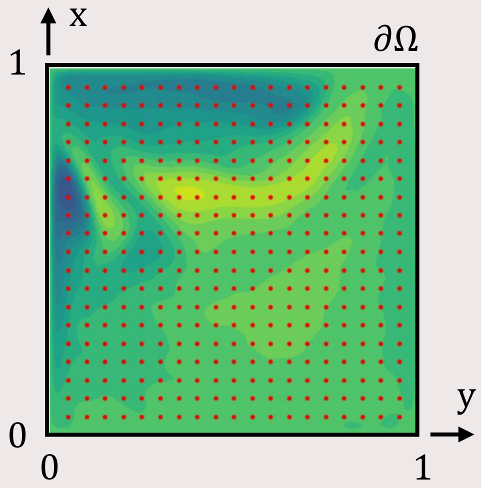
measurement model constructed considering:

- evenly spaced sensor positions $(19\text{x}19)$
- data collection every 10 time-steps



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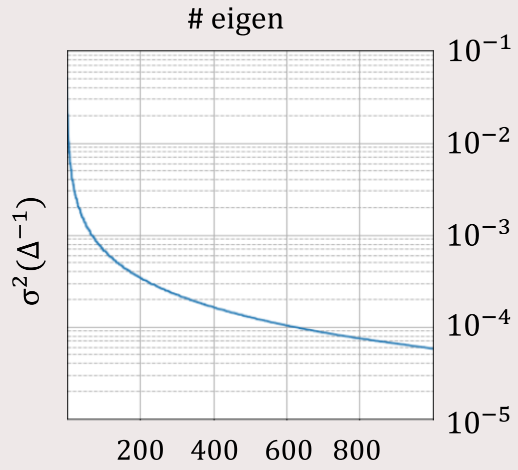
probabilistic model assumes:

- homoscedastic noise $\boldsymbol{\epsilon}_{\mathrm{n}+1} \sim N(0, \sigma^2 \mathbf{I})$ $(\sigma = 10^{-4})$
- normal initial sample distribution $\omega_{0|0} \sim N(0, \Delta^{-1})$



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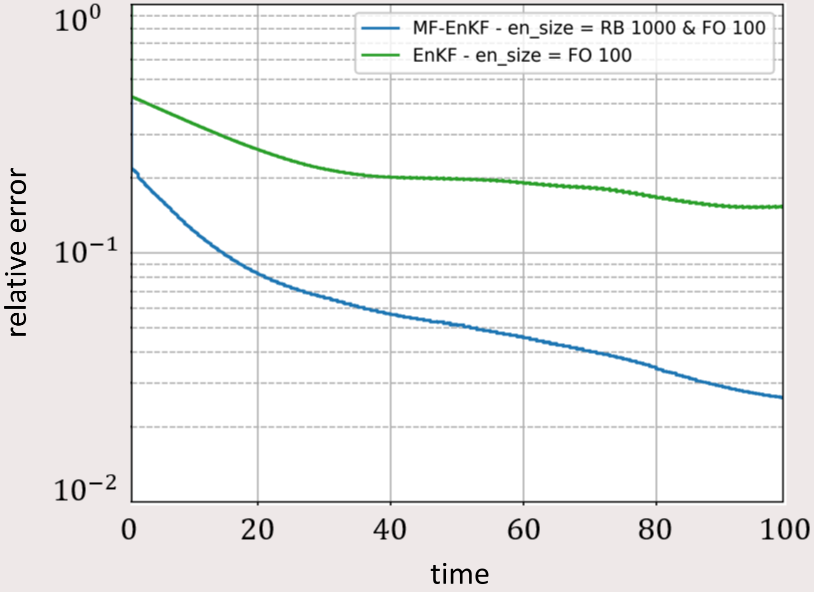

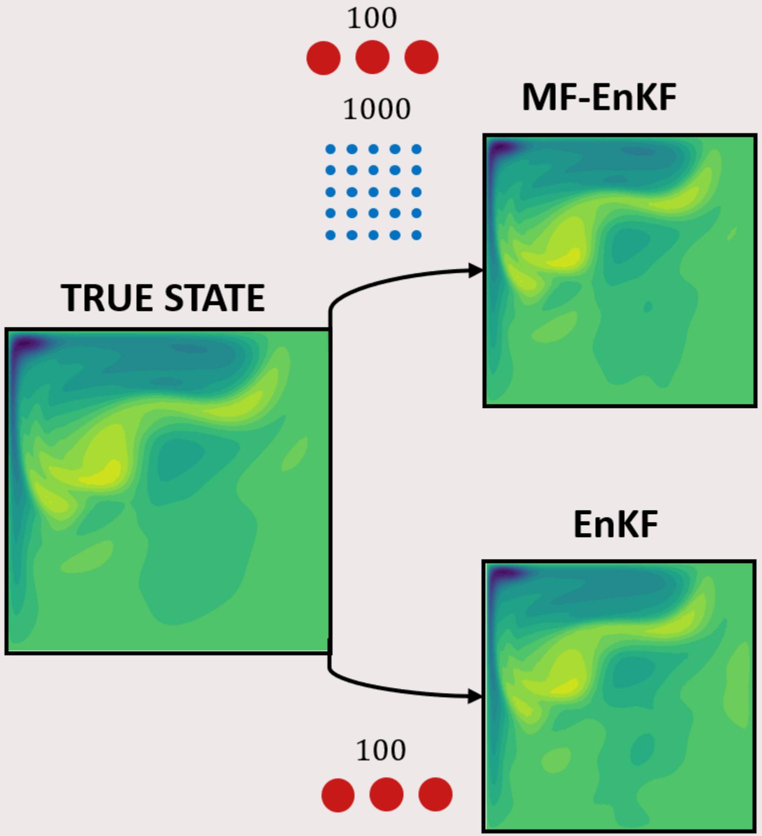

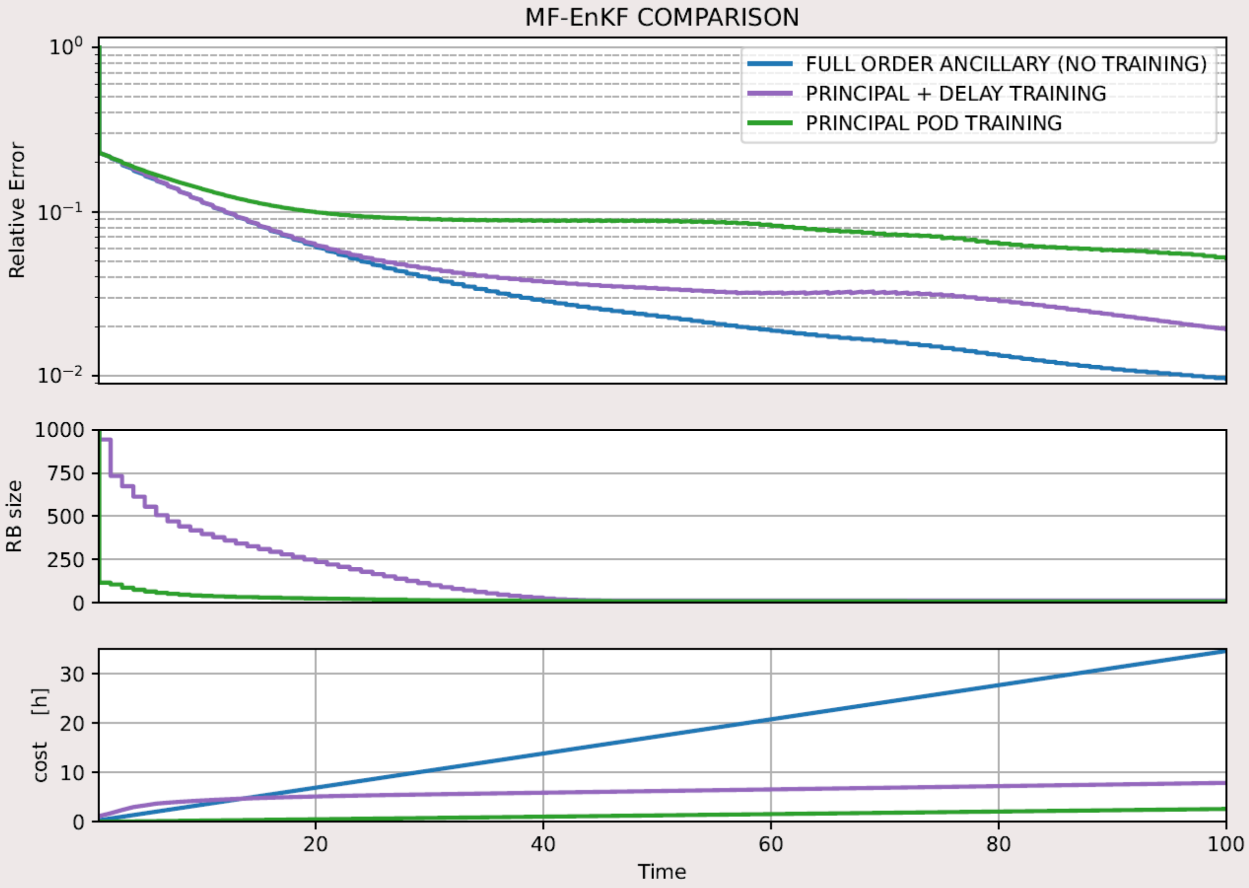
MF-EnKF COMPARISON
FULL ORDER ANCILLARY (NO TRAINING)
PRINCIPAL + DELAY TRAINING
PRINCIPAL POD TRAINING
Relative Error
10^0
10^{-1}
10^{-2}
RB size
1000
750
500
250
0
cost [h]
30
20
10
0
20
40
60
80
100
Time

$\#E_P$: 50

$\#E_A$: 500

CONCLUSIONS

SUMMARY :

- we improved the performances of “adaptive” RB methods for the MF-EnKF
- we performed preliminary tests using a quasi-geostrophic model

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OUTLOOK :

- better investigate the trade off between computational cost and accuracy
- give better theoretical support to the results obtained so far

REFERENCES

[DM22] G. Donoghue and Y. Masayuki. "**A multi-fidelity ensemble Kalman filter with hyperreduced reduced-order models**“. (2022)

[PMS21] A. Popov, C. Mou, A. Sandu, and T. Iliescu. "**A multifidelity ensemble Kalman filter with reduced order control variates.**" (2021)

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[Eve03] G. Evensen. "**The ensemble Kalman filter: Theoretical formulation and practical implementation**". (2003)

[Kal60] R. E. Kalman. "**A new approach to linear filtering and prediction problems**". (1960)

THANKS FOR YOUR ATTENTION!