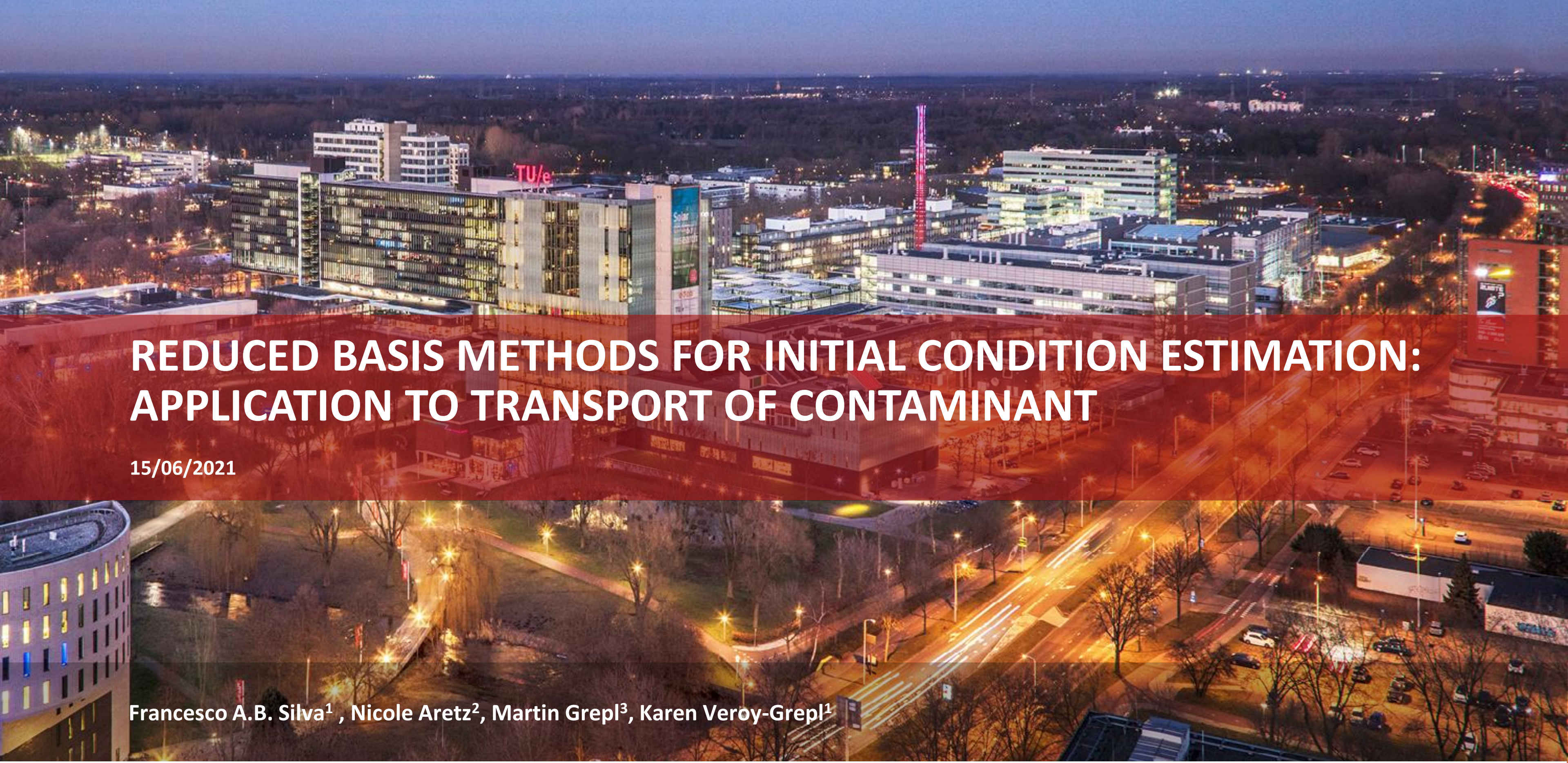

REDUCED BASIS METHODS FOR INITIAL CONDITION ESTIMATION: APPLICATION TO TRANSPORT OF CONTAMINANT

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DATA ASSIMILATION

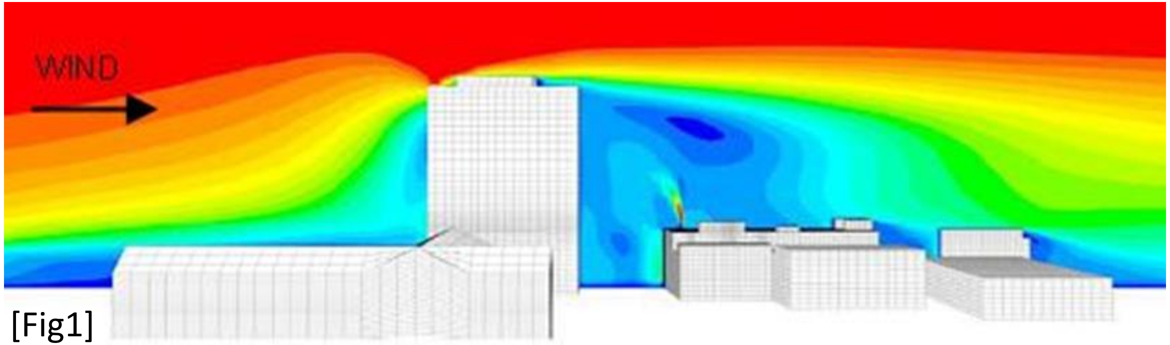
MOTIVATION FOR DATA ASSIMILATION

METEREOLOGY

- Weather forecast
- Air pollution studies
- …

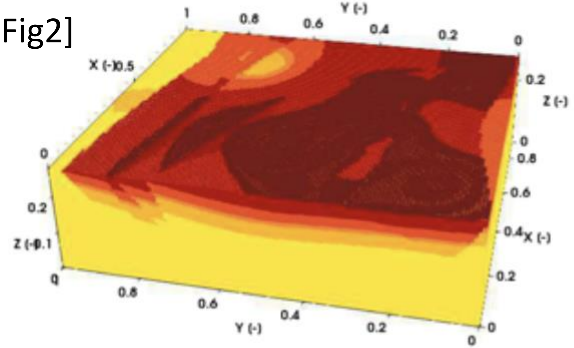
HYDROLOGY

- Ground water management
- Transport of contaminants
- …



[Fig1]

[Fig2]



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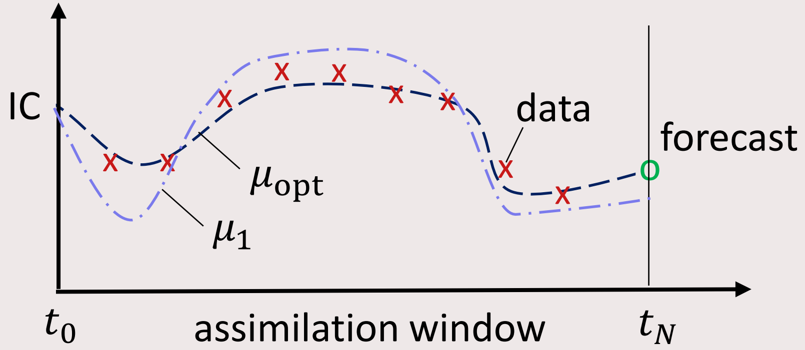
VARIATIONAL DATA ASSIMILATION

GOAL OF THE METHOD

- State forecasting
- Parameter estimation
- IC/BC estimation



STATE OF THE ART

- Reduced Order Modelling [D. N. Daescu et al. 2006]
- A posteriori error bounds [M. Kärcher et al. 2018]
- Optimal sensor location [P. Binev et al 2019]

VARIATIONAL DATA ASSIMILATION – WHAT DO WE NEED?

ON THE MODEL SIDE

- Model of the physical system → Parametric PDE : $A_\mu x = b_\mu$
- A Data Assimilation scheme → 4D-VAR
- Model of the observation process → Linear functional : $Lx = m$

ON THE EXPERIMENTAL SIDE

- Measurements → Experimental data : $d = Lx^{\mathrm{TRUE}} + \epsilon$
- Noise properties → Noise : $\epsilon \sim N(0, \Sigma)$

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PARABOLIC PARAMETRIC PDEs - A SPACE-TIME APPROACH

$$\dot{x}(t) + C_\mu x(t) = g_\mu(t) \quad \text{in } \mathcal{V}', \text{a.e. } t \in I := [0,T]$$
$$x(0) = u \quad \text{in } \mathcal{H}$$

from which the variational formulation:

$$\int_I \langle \dot{x}(t) + C_\mu x(t), \eta(t) \rangle_{\mathcal{H}} \, dt = \int_I \langle g_\mu(t), \eta(t) \rangle_{\mathcal{H}} dt \qquad \forall \ \eta(t) \in L^2(I,\mathcal{V})$$
$$\langle x(0), \xi \rangle_{\mathcal{H}} = \langle u, \xi \rangle_{\mathcal{H}} \qquad \forall \ \xi \in \mathcal{H}$$

ψ (test functions $\eta(t)$, ξ) $\quad \mathcal{Y}$ (spaces $L^2(I,\mathcal{V})$, $\mathcal{H}$)

that can be rewritten as:

$$a_\mu(x,\psi) = f_\mu^{\text{bk}}(\psi) + b_\mu(u,\psi) \quad \forall \psi \in \mathcal{Y}$$

$$x \in \mathcal{X} := L^2(I,\mathcal{V}) \cap H^1(I,\mathcal{V}'), \ u \in \mathcal{U} \subset \mathcal{H}$$

SPACE-TIME WEAK MODEL

STRONG 4D-VAR – AN OPTIMAL CONTROL PROBLEM

$$\min_{u\in\mathcal{U}} \mathcal{J}(u \mid \mu, d) := \boxed{\frac{1}{2}\|Lx - d\|_{\mathcal{Z}}^2} + \boxed{\frac{\lambda}{2}\|u\|_{\mathcal{U}}^2} \quad s.t. \quad \boxed{a_\mu(x,\psi) = f_\mu^{\mathrm{bk}}(\psi) + b_\mu(u,\psi) \quad \forall\psi \in \mathcal{Y}}$$

MISFIT STABILIZATION MODEL

where:

$d \in \mathcal{Z}$: Measurements

$u \in \mathcal{U}$: Initial conditions

$x \in \mathcal{X}$: Bochner state

λ : depends on the trust we have in the prior knowledge !

we assume:

$d = Lx^{\mathrm{TRUE}} + \epsilon$ *with noise* $\epsilon \sim N(0, \Sigma)$

STABILITY OF THE MINIMIZATION PROBLEM

Given this structure of the minimization problem, we can prove that:

$$\|u(d+\epsilon)-u(d)\|_{\mathcal{U}} \leq \frac{\|\epsilon\|_{\mathcal{Z}}}{\sqrt{2\lambda+\beta_L^2(\mu)\underline{\eta}^2(\mu)}}$$

STABILITY BOUND

Where:

WE CAN ACT ON THIS

$$\beta_L(\mu) := \inf_{u\in\mathcal{U}} \frac{\|Lx(u|\mu)\|_{\mathcal{Z}}}{\|u\|_{\mathcal{U}}}$$ Observability ← depends on the experimental design

$$\underline{\eta}(\mu) := \inf_{u\in\mathcal{U}} \frac{\|x(u|\mu)\|_{\mathcal{X}}}{\|u\|_{\mathcal{U}}}$$ Sensibility to ICs ← depends on the described physics

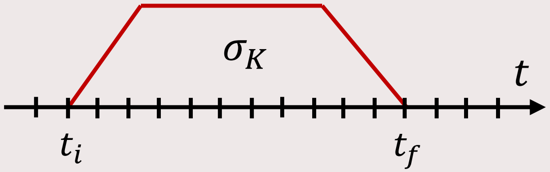
MEASUREMENT MODEL - MULTIPLE DATA COLLECTION

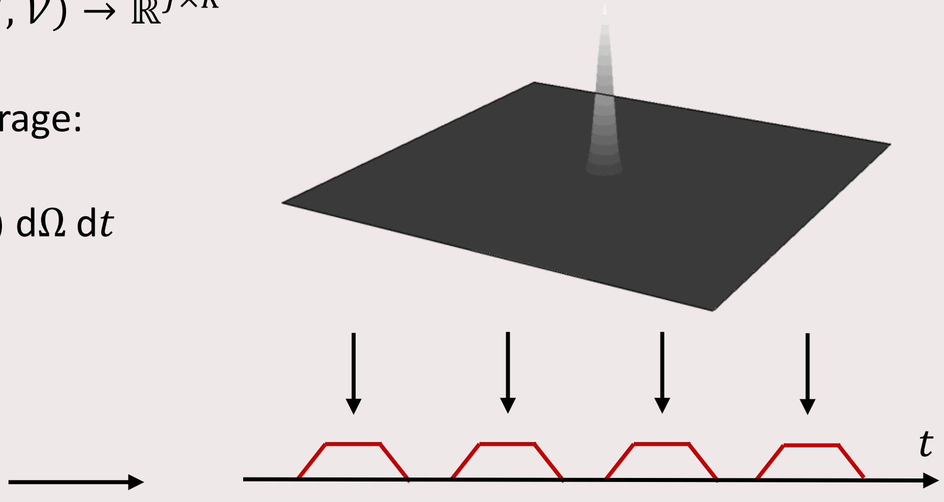
We describe the whole measuring process as a collection of sensors $\ell_{J,K}(\cdot)$:

$$L(\xi) = \left[\ell_{1,1}(\xi), \dots, \ell_{J,K}(\xi)\right] \quad : \quad L^2(I, \mathcal{V}) \to \mathbb{R}^{J \times K}$$

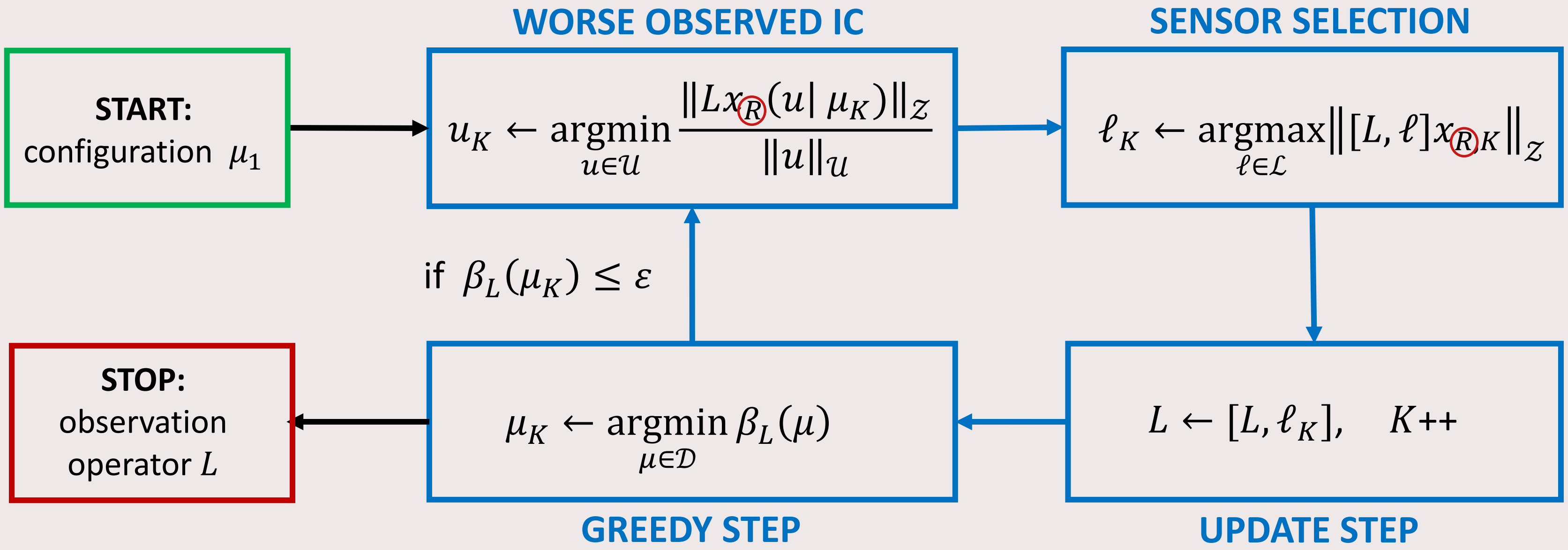
Each sensor performs the weighted average:

$$\ell_{J,K}(\xi) = \int_I \int_\Omega \xi(\boldsymbol{r}, t) \cdot \rho_J(\boldsymbol{r}) \cdot \sigma_K\ (\mathrm{t})\ \mathrm{d}\Omega\ \mathrm{d}t$$





IMPROVE SYSTEM OBSERVABILITY - GREEDY OMP



ADVECTION-DISPERSION PROBLEM

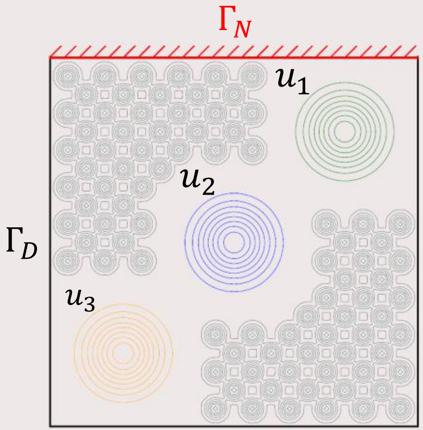
ADVECTION-DISPERSION PROBLEM

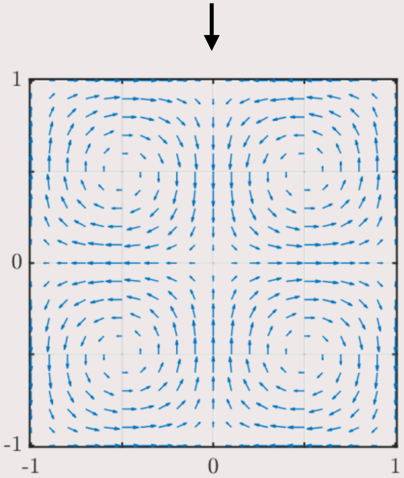
$$\frac{\partial x}{\partial t} - \frac{\Delta x(t)}{\mu} + \boldsymbol{v} \cdot \boldsymbol{\nabla} x(t) = 0 \quad \text{on } \Omega := (-1, +1)^2 \qquad \text{with} \qquad \boldsymbol{v} = \begin{bmatrix} +\sin(\pi x_1)\cos(\pi x_2) \\ -\cos(\pi x_1)\sin(\pi x_2) \end{bmatrix}$$

$$x(0) = u$$

We consider:

- 3 initial conditions
- 88 sensor locations
- 160 activations per sensor
- $t \in (0, 8)$
- $\mu \in [10, 50]$





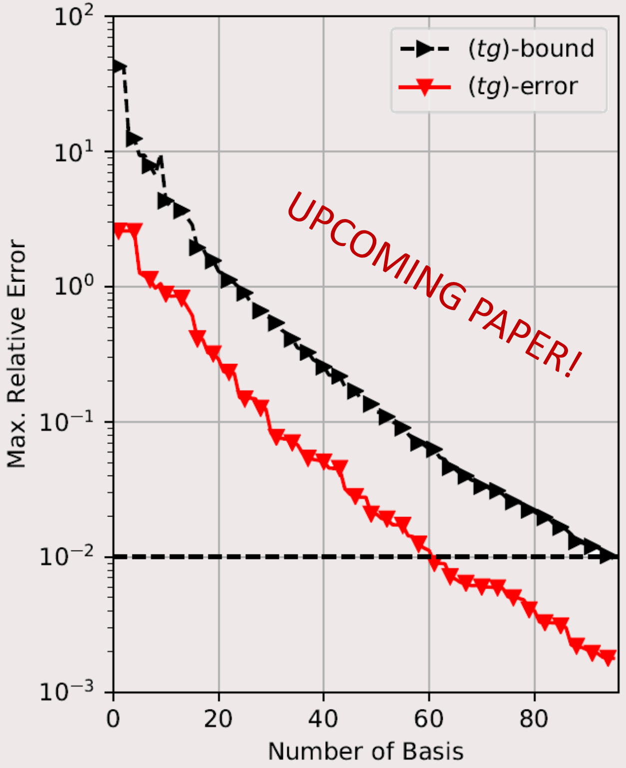
MODEL ORDER REDUCTION

Employing the weak-greedy-POD approach we construct a Reduced Basis space of size 96 :

dofs spatial discretization = 10100 (P2-P2 G)
dofs time discretization = 801 (P1-P0 PG)

training set size = 3 x 80
training time = 315s

The effectivity of the bound is independent from the space dimension and doesn't exceed a factor 10



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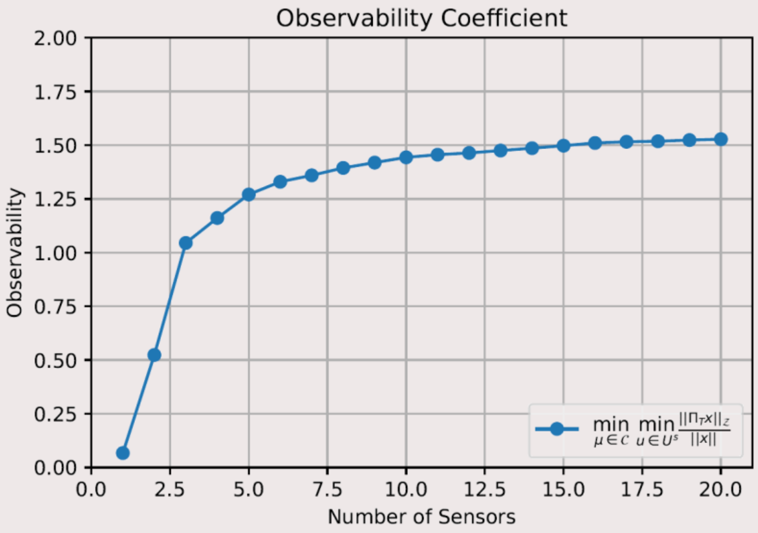
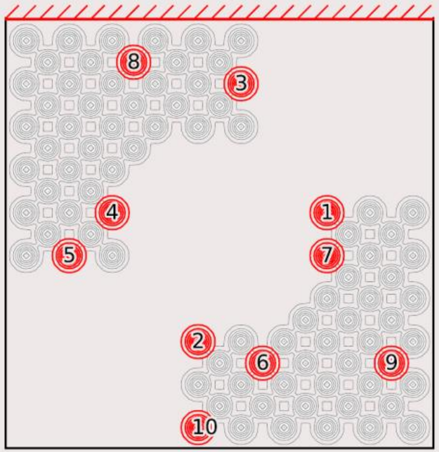
SENSOR SELECTION

Thanks to the RB model we can efficiently select nearly optimal sensor locations





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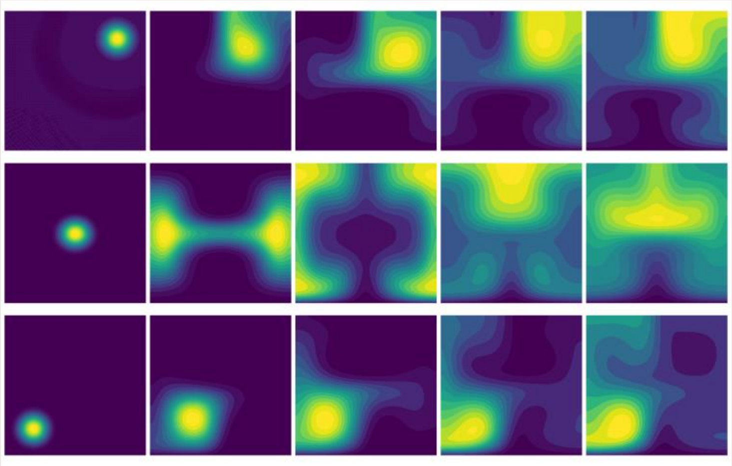
VISUALIZATION ON THE SOLUTIONS

$\mu = 25.25$



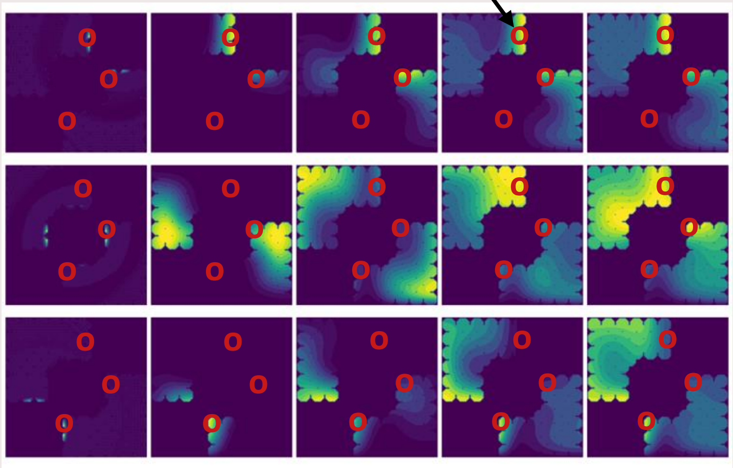
t: 0.05 1.05 2.05 3.05 4.05

absolute value decreases

$\mathcal{L}$

first 3 sensors

$\mu = 25.25$



t: 0.05 1.05 2.05 3.05 4.05

absolute value decreases

CONCLUSIONS

WHAT HAVE WE ALREADY DONE?

- We developed an effective error bound for coercive parabolic problems
- We implemented a greedy-POD algorithm for the reduction of the problem
- We extended the greedy-OMP algorithm to space-time states

WHAT ARE WE STILL WORKING ON?

- We are testing a 4D-VAR solver based on a Lagrange optimization scheme
- We are improving an error bound for the adjoint problem
- We are developing a gradient free scheme to solve the 4D-VAR problem

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SELECTED RELATED LITERATURE

[P. Binev et al 2019] : Greedy Algorithms for Optimal Measurements Selection in State Estimation Using Reduced Model

[M. Kärcher et al. 2018] : Reduced basis approximation and a-posteriori error bounds for 4D-Var data assimilation

[D. N. Daescu et al. 2006] : Efficiency of a POD-based reduced second-order adjoint model in 4D-Var data assimilation

[N. Aretz et al. 2019] : 3D-VAR for parameterized partial differential equations: a certified reduced basis approach

[Fig1] : Z. Huijbregts et al., CFD simulation of pollutant gas dispersion in downtown Montreal

[Fig2] : Denise Degen et al, Certified Reduced Basis Method in Geosciences

THANK YOU FOR YOUR ATTENTION!

QUESTION TIME!